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Optional and predictable projection of a normal integrand

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Työssä yleistetään mitallisen integrandin optionaalisen ja ennustettavan projektion olemassaolo- ja yksikäsitteisyystulos diskreetistä ajasta jatkuvaan aikaan sekä annetaan riittävät ehdot integrandin normaaliuden säilymiselle.

Avainsanat: optionaalinen projektio, ennustettava projektio, normaali-integrandi

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ABSTRACT OF THE
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This thesis discusses on the optional and predictable projection of a measurable integrand. We generalize the existence and uniqueness result from discrete time to continuous time and give sufficient conditions for the preservation of normality.

Keywords: optional projection, predictable projection, normal integrand

Contents

1 Introduction

Early probability theory was a study of random variables, which are measurable mappings from a probability space Ω to the real numbers. A central concept in the study of random variables is that of a conditional expectation. Its measure-theoretic definition is given in the equation (3) at the page 2. The conditional expectation tells us the expected value of a random variable in light of the additional information, and it can interpreted as a *projection* of the random variable on that information.

Over time the common interest of research shifted from random variables to stochastic processes. A stochastic process is a sequence $v = (v_t)$, $t \in \mathbb{R}_+$, where each v_t is a random variable. The index set $\mathbb{R}_+$ is often interpreted as a time, and sometimes it is convenient to think of a stochastic process as a time-dependent random variable, and sometimes simply as a function on $\Omega \times \mathbb{R}_+$. On $\Omega \times \mathbb{R}_+$, there are two σ-algebras, that are of particular importance, the one generated by adapted right-continuous processes and the other generated by adapted left-continuous processes, namely the optional and the predictable σ-algebra.

Study of these two σ-algebras gives a natural rise to so-called stopping times, which are measurable mappings to the index set. A projection of process is what one expects to have when the process stops. A classical result from the general theory of stochastic processes states that, under fairly general measurability and integrability conditions, there exist an optional and a predictable projection of a stochastic process and that these projections are unique (see Theorem 4.37).

Consider next a real valued function f on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d$. If f is measurable, then it can be understood in various ways. For example, we can look at it as a sequence $f = (f_t)$, $t \in \mathbb{R}_+$, where each f_t is a random variable depending on the parameter $x \in \mathbb{R}^d$, or alternatively, each f_t is a random real valued function on $\mathbb{R}^d$. One encounters these kind of functions, e.g., in stochastic optimization, and motivated by that, we allow f to take values $+\infty$ and $-\infty$. In stochastic optimization, this is a nice feature, since it enables to embed the parameter constraints in the function itself. See further discussion of this stochastic optimization framework from Rockafellar [1976].

We will call f described above an integrand, and if it satisfies additional measurability and continuity assumptions, we will call it normal. The in-detail definition of normality is given in Chapter 5. In Chapter 5, we will generalize aforementioned Theorem 4.37 for measurable integrands, and show that, under certain integrable minorant condition, the optional and predictable projection of a normal integrand are normal integrands as well. These are new results.

The main references of this thesis are the book *Semimartingale Theory and Stochastic Calculus* by He, Wang and Yang and the voluminous book *Probability and Potential* by Dellacherie and Meyer. Excluding few exceptions, all of the results presented in the following three chapters can be found from both of these books, and for those results, which can be found from neither of the books, an explicit reference is given. We follow the presentation of the first book for the most parts. In addition to the books, I want to mention the blog called *Almost Sure* by George Lowther, which found to be a valuable complementary reading.

2 Preliminaries

In this chapter we recall some facts from probability theory. We assume the reader to be familiar with the basic results such as *Markov's inequality*, *Fatou's lemma* and the *monotone convergence theorem*.

2.1 The Generalization of Conditional Expectation

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $\overline{\mathbb{R}}$ denote the extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$, and f be a measurable extended real valued mapping on Ω. We make the convention $+\infty + (-\infty) = +\infty$ on the extended real line. Let $\mathbf{S}$ denote the class of simple real valued functions on Ω. Put $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$. Consider the following Lebesgue integral

$$\begin{aligned}\int f(\omega)\mathbf{P}(d\omega) &= \int f^+(\omega)\mathbf{P}(d\omega) - \int f^-(\omega)\mathbf{P}(d\omega) \\ &= \sup_{\substack{g^+ \le f^+ \\ g^+ \in \mathbf{S}}} \int g^+(\omega)\mathbf{P}(d\omega) - \sup_{\substack{g^- \le f^- \\ g^- \in \mathbf{S}}} \int g^-(\omega)\mathbf{P}(d\omega),\end{aligned} \tag{1}$$

with the convention

$$\int f^+(\omega)\mathbf{P}(d\omega) = \infty \implies \int f(\omega)\mathbf{P}(d\omega) = \infty. \tag{2}$$

We will use the terminology of probability theory. A measurable mapping is called a *random variable*, and often denoted by x, and the integral (1) of x with the convention (2) is called the *expectation of* x, denoted by $\mathbf{E}[x]$.

The following definition is due to Andrei Kolmogorov who gave the first axiomatic treatment of probability; Kolmogorov [1933]. Let x be a measurable random variable and $\mathcal{G}$ be an arbitrary sub-σ-algebra of $\mathcal{F}$. Then a $\mathcal{G}$-measurable random variable y, satisfying

$$\int_G x(\omega)\mathbf{P}(d\omega) = \int_G y(\omega)\mathbf{P}(d\omega) \text{ for every } G \in \mathcal{G}, \tag{3}$$

is called the *conditional expectation of* x. The conditional expectation of x is denoted by $\mathbf{E}[x|\mathcal{G}]$. Note that since we allow the integral in (3) take values $-\infty$ and $+\infty$, the conditional expectation is well defined for all random variables, i.e., it exists and is unique up to a null set. However, this is a trade-off. The cost of being able to define conditional expectation for all random variables is that the conditional expectation is only a sublinear operator in the class: it is not linear without an additional integrability assumption.

A random variable x is said to be *integrable*, if $\mathbf{E}[|x|] < \infty$ and a family of random variables $\{x_i\}$, $i \in I$, is said to be *uniformly integrable*, if

$$\lim_{c \to \infty} \sup_{i \in I} \mathbf{E}[|x_i| 1_{\{|x_i| > c\}}] = 0.$$

Proposition 2.1. *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be a probability space,* x *an integrable random variable, and* $\{\mathcal{G}_i\}$*,* $i \in I$*, a family of sub-*σ*-algebras of* $\mathcal{F}$*. Then* $\{\mathbf{E}[x|\mathcal{G}_i]\}$ *is a uniformly integrable family.*

Proof. Set $y_i = \mathbf{E}[|x||\mathcal{G}_i]$. By Markov's inequality, for every $c > 0$, we have

$$\mathbf{P}(y_i \geq c) \leq \frac{1}{c}\mathbf{E}[y_i] = \frac{1}{c}\mathbf{E}[|x|],\ i \in I,$$

and, for every $\delta > 0$, we have

$$\begin{aligned}\int_{\{y_i > c\}} y_i \mathbf{P}(d\omega) = \int_{\{y_i > c\}} |x|\mathbf{P}(d\omega) &\leq \delta \mathbf{P}(y_i > c) + \int_{\{|x|>\delta\}} |x|\mathbf{P}(d\omega) \\ &\leq \frac{\delta}{c}\mathbf{E}[|x|] + \int_{\{|x|>\delta\}} |x|\mathbf{P}(d\omega),\ i \in I.\end{aligned}$$

For any given $\epsilon > 0$, we may choose $\delta > 0$ such that $\int_{\{|x|>\delta\}} |x|\mathbf{P}(d\omega) \leq \epsilon/2$. When $c \geq \frac{2\delta}{\epsilon}\mathbf{E}[|x|]$, we have $\int_{\{y_i>c\}} y_i\mathbf{P}(d\omega) \leq \epsilon$ for $i \in I$, i.e., the family $\{y_i\}$ is uniformly integrable. □

We will use an upward pointing arrow $\uparrow$ to denote *upward monotone convergence*, which for the sets, say A_n, $n \in \mathbb{N}$, and A, means that, for every n, $A_n \subset A_{n+1}$ and $A = \bigcup_{n=1}^{\infty} A_n$. A downward pointing arrow $\downarrow$ is a symbol for *downward monotone convergence*, which, for the sets, means that, for every n, $A_n \supset A_{n+1}$ and $A = \bigcap_{n=1}^{\infty} A_n$. If A_n, $n \in \mathbb{N}$, and A are real numbers or (extended) real valued functions instead of sets, we replace inclusion, union and intersection with inequality, supremum and infimum, respectively, in the previous definitions.

A random variable x is said to be *σ-integrable with respect to $\mathcal{G}$*, if there exists $(\Omega_n) \subset \mathcal{G}$, $1_{\Omega_n} \uparrow 1_\Omega$ a.s., such that $x1_{\Omega_n}$ is integrable for each n. The suffix *a.s.* is an abbreviation for *almost surely*, a familiar notion of probability theory, which means that the relation holds with probability one.

Proposition 2.2. *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be a probability space,* $\mathcal{G}$ *a sub-*σ*-algebra of* $\mathcal{F}$*, and* x *a random variable. If* x *is* σ*-integrable w.r.t.* $\mathcal{G}$*, then* $\mathbf{E}[|x||\mathcal{G}] < \infty$ *a.s..*

Proof. We may assume without a loss of generality that x is non-negative. Suppose $(\Omega_n) \subset \mathcal{G}$, $1_{\Omega_n} \uparrow 1_\Omega$ a.s. and each $x1_{\Omega_n}$ is integrable. Put

$$y_n = \mathbf{E}[x1_{\Omega_n}|\mathcal{G}].$$

Then $y_{n+1}1_{\Omega_n} = y_n$ a.s., $y_n \uparrow y$ a.s., where the limit y is a finite $\mathcal{G}$-measurable random variable. For $G \in \mathcal{G}$ we have

$$\mathbf{E}[x1_G] = \lim_n \mathbf{E}[x1_G1_{\Omega_n}] = \lim_n \mathbf{E}[y_n1_G] = \mathbf{E}[y1_G],$$

i.e., $y = \mathbf{E}[x|\mathcal{G}]$. □

The conditional expectation of a σ-integrable random variable has the smoothing properties of the conditional expectation of an integrable random variable, some of which are listed in the following Proposition 2.3.

Proposition 2.3. *Assume $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space, $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, and x and y are two random variables such that x is σ-integrable w.r.t. $\mathcal{G}$. Then we have the following smoothing properties:*

(a) *If y is $\mathcal{G}$-measurable and finite, then xy is σ-integrable w.r.t. $\mathcal{G}$, and*

$$\mathbf{E}[xy|\mathcal{G}] = y\mathbf{E}[x|\mathcal{G}] \ \textit{a.s.}. \tag{4}$$

(b) *If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{F}$ such that $\mathcal{G} \subset \mathcal{H}$, then $\mathbf{E}[x|\mathcal{H}]$ is σ-integrable w.r.t. $\mathcal{G}$, and*

$$\mathbf{E}[x|\mathcal{G}] = \mathbf{E}[\mathbf{E}[x|\mathcal{H}]|\mathcal{G}] \ \textit{a.s.}. \tag{5}$$

(c) *If $A \in \mathcal{G}$ and $x = y1_A$, then $x1_A$ is σ-integrable w.r.t. $\mathcal{G}' = \sigma\{A \cap G : G \in \mathcal{G}\}$, and*

$$\mathbf{E}[x1_A|\mathcal{G}] = \mathbf{E}[x1_A|\mathcal{G}'] \ \textit{a.s.}. \tag{6}$$

Proof. (a) Suppose $(A_n) \subset \mathcal{G}$, $A_n \uparrow \Omega$ a.s. and each $x1_{A_n}$ is integrable. Set $B_n = \{|y| \leq n\}$. Then $(B_n) \subset \mathcal{G}$ and $B_n \uparrow \Omega$. Put $\Omega_n = A_n \cap B_n$. Then $(\Omega_n) \subset \mathcal{G}$, $1_{\Omega_n} \uparrow 1_\Omega$ a.s. and each $xy1_{\Omega_n}$ is integrable, i.e., xy is σ-integrable. w.r.t. $\mathcal{G}$. By Proposition 2.2, we have

$$\mathbf{E}[xy|\mathcal{G}]1_{\Omega_n} = \mathbf{E}[xy1_{\Omega_n}|\mathcal{G}] = y1_{\Omega_n}\mathbf{E}[x|\mathcal{G}] \text{ a.s.}. \tag{7}$$

Letting $n \to \infty$ in (7) yields (4).

(b) Suppose $(\Omega_n) \subset \mathcal{G}$, $1_{\Omega_n} \uparrow 1_\Omega$ a.s. and each $x1_{\Omega_n}$ is integrable. By Proposition 2.2, we have that $\mathbf{E}[x|\mathcal{H}]1_{\Omega_n}$ is integrable, and therefore, $\mathbf{E}[x|\mathcal{H}]$ is σ-integrable w.r.t. $\mathcal{G}$. Moreover, we have

$$\mathbf{E}[x1_{\Omega_n}|\mathcal{G}] = \mathbf{E}[\mathbf{E}[x1_{\Omega_n}|\mathcal{H}]|\mathcal{G}] \text{ a.s.}.$$

By (a), we obtain

$$\mathbf{E}[x|\mathcal{G}]1_{\Omega_n} = \mathbf{E}[x1_{\Omega_n}|\mathcal{G}] = \mathbf{E}[\mathbf{E}[x1_{\Omega_n}|\mathcal{H}]|\mathcal{G}] = \mathbf{E}[\mathbf{E}[x|\mathcal{H}]1_{\Omega_n}|\mathcal{G}] = \mathbf{E}[\mathbf{E}[x|\mathcal{H}]|\mathcal{G}]1_{\Omega_n} \text{ a.s.}. \tag{8}$$

Letting $n \to \infty$ in (8) yields (5).

(c) It is clear that $x1_A$ is σ-integrable w.r.t. $\mathcal{G}'$, and by Proposition (a), we have

$$\mathbf{E}[x1_A|\mathcal{G}] = \mathbf{E}[x1_A|\mathcal{G}]1_A \text{ a.s.}.$$

Hence, $\mathbf{E}[x1_A|\mathcal{G}]$ can be considered as a $\mathcal{G}'$-measurable random variable, and since $\mathcal{G}' \subset \mathcal{G}$, by (b), we have

$$\mathbf{E}[x1_A|\mathcal{G}'] = \mathbf{E}[\mathbf{E}[x1_A|\mathcal{G}']|\mathcal{G}] = \mathbf{E}[x1_A|\mathcal{G}] \text{ a.s.},$$

i.e., (6) holds. $\square$

2.2 Monotone Class Theorems

Let F be a set, and $\mathcal{C}$ a collection of subsets of F. We say that $\mathcal{C}$ is a *class* on F. If a class $\mathcal{C}$ contains empty set $\emptyset$, we say that $\mathcal{C}$ is a *paving* on F. If in addition $\mathcal{C}$ is closed under taking a complement and a finite (resp. countable) intersection, we say that $\mathcal{C}$ is an *algebra* (resp. *σ-algebra*) on F.

The σ-algebra generated by a function $f : \Omega \to \mathbb{R}$ is defined as $\sigma(f) = \{f^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra of $\mathbb{R}$. The σ-algebra generated by a class $\mathcal{C}$ is denoted by $\sigma(\mathcal{C})$. The notions closely related to the σ-algebra generated by a class are $\mathcal{C}_\sigma = \{\bigcup_{k=1}^{\infty} A_k : A_k \in \mathcal{C}\}$ and $\mathcal{C}_\delta = \{\bigcap_{k=1}^{\infty} A_k : A_k \in \mathcal{C}\}$. We use the notation $\mathcal{C}_{\sigma\delta}$ for $(\mathcal{C}_\sigma)_\delta$.

A class $\mathcal{C}$ on F is called a *π-class*, if it is closed under taking intersection, and a *monotone class*, if it is closed under monotone convergence; $A_n \in \mathcal{C}, A_n \uparrow A$ or $A_n \downarrow A \implies A \in \mathcal{C}$. A monotone class $\mathcal{C}$ is called a *λ-class*, if $F \in \mathcal{C}$ and $A_n \in \mathcal{C}, A \subset B \implies B\backslash A \in \mathcal{C}$. Obviously, a λ-class is a monotone class and an algebra is a π-class. If $\mathcal{C}$ is both a π-class and a λ-class, or both an algebra and a monotone class, then $\mathcal{C}$ is a σ-algebra.

The monotone convergence of sets corresponds to the continuity in measure. The following two propositions emphasize the fact.

Proposition 2.4. *Let $(F, \mathcal{F}, \mu)$ be a finite measure space and $\mathcal{C}$ be an algebra generating $\mathcal{F}$. Then for any $A \in \mathcal{F}$ we have*

$$\mu(A) = \sup\{\mu(B) : B \in \mathcal{C}_\delta, B \subset A\} = \inf\{\mu(C) : C \in \mathcal{C}_\sigma, A \subset C\}. \tag{9}$$

Proof. Put

$$\mathcal{G} = \{A \in \mathcal{F} : A \text{ satisfies } (9)\}.$$

We have $\mathcal{C} \subset \mathcal{G} \subset \mathcal{F}$ with $\mathcal{F} = \sigma(\mathcal{C})$. It suffices to show that $\mathcal{G}$ is a σ-algebra.

Since $\mathcal{C}_\sigma = \{A : A^c \in \mathcal{C}_\delta\}$, we have $A \in \mathcal{G} \implies A^c \in \mathcal{G}$. Let $(A_n) \subset \mathcal{G}$, $A_n \uparrow A$. For any given $\epsilon > 0$, we may choose n such that $\mu(A\backslash A_n) < \epsilon/2$ and $B \in \mathcal{C}_\delta$, $B \subset A_n$ such that $\mu(A\backslash B) < \epsilon/2$. Then $B \subset A$ and $\mu(B) > \mu(A) - \epsilon$. On the other hand, if for each n, we take $C_n \in \mathcal{C}_\sigma$, $A_n \subset C_n$ such that $\mu(C_n\backslash A_n) < \epsilon/2^n$ and put $C = \bigcup_{n=1}^{\infty} C_n$. We have $C \in \mathcal{C}_\sigma$, $A \subset C$ and $\mu(C\backslash A) < \epsilon$. This means that $A \in \mathcal{G}$. We have shown that $\mathcal{G}$ is an algebra and a monotone class. This means that $\mathcal{G}$ is a σ-algebra, and thus $\mathcal{G} = \mathcal{F}$. ∎

Proposition 2.5. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and $\mathcal{H}$ be a non-empty collection of random variables. Then there exists an extended real valued random variable y such that*

(a) $x \le y$ a.s. for all $x \in \mathcal{H}$,

(b) If $x \le \tilde{y}$ a.s. for all $x \in \mathcal{H}$, then $y \le \tilde{y}$ a.s..

Moreover, there exists a sequence $(x_n) \subset \mathcal{H}$ such that $y = \bigvee_n x_n$. The random variable y is called the essential supremum of $\mathcal{H}$ and denoted by $\operatorname{ess\,sup} \mathcal{H}$*.*

Proof. We may assume that the family $\mathcal{H}$ is bounded. Otherwise, we may consider the family $\{\arctan(x) : x \in \mathcal{H}\}$ instead. Furthermore, we may assume that the family $\mathcal{H}$ is closed under the operation $\vee$. Let $(x_n) \subset \mathcal{H}$ be a monotone increasing sequence such that $\lim_n \mathbf{E}[x_n] = \sup_{x\in\mathcal{H}} \mathbf{E}[x]$. Put $y = \bigvee_n x_n$. It is clear that y satisfies the property (b). Let us show (a). For every $x \in \mathcal{H}$ put $\tilde{x}_n = x \vee x_n$. Then $(\tilde{x}_n) \subset \mathcal{H}$ is a monotone increasing sequence and $\lim_n \tilde{x}_n = x \vee y$. We have

$$\mathbf{E}[x \vee y] = \lim_n \mathbf{E}[\tilde{x}_n] \leq \sup_{x\in\mathcal{H}}[x_n] = \mathbf{E}[y].$$

Because $y \leq x \vee y$, we have $y = x \vee y$ a.s., i.e., $x \leq y$ a.s.. □

Theorem 2.6. (Monotone class theorem for sets). *Let $\mathcal{C}$ and $\mathcal{E}$ be two classes on F, and $\mathcal{C} \subset \mathcal{E}$.*

(a) *If $\mathcal{E}$ is a λ-class and $\mathcal{C}$ is a π-class, then $\sigma(\mathcal{C}) \subset \mathcal{E}$.*

(b) *If $\mathcal{E}$ is a monotone class and $\mathcal{C}$ is an algebra, then $\sigma(\mathcal{C}) \subset \mathcal{E}$.*

Proof. (a) An intersection of an arbitrary collection of λ-classes is a λ-class. Let $\mathcal{E}_0$ be the intersection of all λ-classes containing $\mathcal{C}$. Put

$$\mathcal{E}_1 = \{A \in \mathcal{E}_0 : \forall B \in \mathcal{C},\ A \cap B \in \mathcal{E}_0\}.$$

Then $\mathcal{E}_1$ is λ-class containing $\mathcal{C}$. Hence, $\mathcal{E}_0 = \mathcal{E}_1$. Put

$$\mathcal{E}_2 = \{A \in \mathcal{E}_0 : \forall B \in \mathcal{E}_0,\ A \cap B \in \mathcal{E}_0\}.$$

Equally, $\mathcal{E}_2$ is λ-class containing $\mathcal{C}$. Hence, $\mathcal{E}_0 = \mathcal{E}_2$, and $\mathcal{E}_0$ is a π-class. This means that $\mathcal{E}_0$ is a σ-algebra, and $\sigma(\mathcal{C}) \subset \mathcal{E}_0 \subset \mathcal{E}$.

(b) An intersection of an arbitrary collection of monotone classes is a monotone class. Let $\mathcal{E}_0$ be the intersection of all monotone classes containing $\mathcal{C}$. As above, one can show that $\mathcal{E}_0$ is a π-class as well. Put

$$\mathcal{E}_1 = \{A \in \mathcal{E}_0 : A^c \in \mathcal{E}_0\}.$$

Then $\mathcal{E}_1$ is a monotone class containing $\mathcal{C}$. Hence $\mathcal{E}_0 = \mathcal{E}_1$, and $\mathcal{E}_0$ is an algebra. This means that $\mathcal{E}_0$ is a σ-algebra, and $\sigma(\mathcal{C}) \subset \mathcal{E}_0 \subset \mathcal{E}$. □

Corollary 2.7. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, x and y integrable random variables, and $\mathcal{C}$ a π-class such that $\Omega \in \mathcal{C}$ and $\mathcal{C} \subset \mathcal{F}$. If $\mathbf{E}[x1_A] = \mathbf{E}[y1_A]$ for every $A \in \mathcal{C}$, then*

$$\mathbf{E}[x|\sigma(\mathcal{C})] = \mathbf{E}[y|\sigma(\mathcal{C})] \text{ a.s.}. \tag{10}$$

Proof. Put

$$\mathcal{G} = \{A \in \mathcal{F} : \mathbf{E}[x1_A] = \mathbf{E}[y1_A]\}.$$

Then $\mathcal{G}$ is λ-class, and $\mathcal{C} \subset \mathcal{G}$. By Theorem 2.6, $\sigma(\mathcal{C}) \subset \mathcal{G}$, i.e., (10) holds. □

Theorem 2.8. (Monotone class theorem for extended real valued functions). *Let $\mathcal{C}$ be a π-class on F that contains F, and $\mathcal{V}$ be a collection of extended real valued functions on F. If the following conditions are satisfied:*
(a) $A \in \mathcal{C} \implies 1_A \in \mathcal{V}$,
(b) $\alpha, \beta \in \mathbb{R}_+,\ f, g \geq 0,\ f, g \in \mathcal{V} \implies \alpha f + \beta g \in \mathcal{V}$,
(c) $f_n \in \mathcal{V},\ 0 \leq f_n \uparrow f \implies f \in \mathcal{V}$,
(d) $f, g \geq 0,\ f, g \in \mathcal{V} \implies f - g \in \mathcal{V}$,
then $\mathcal{V}$ contains all $\sigma(\mathcal{C})$-measurable extended real valued functions on F.

Proof. Put $\mathcal{E} = \{A \subset F : 1_A \in \mathcal{V}\}$. Then $\mathcal{E}$ is a λ-class and $\mathcal{C} \in \mathcal{E}$ by the assumption. From Theorem 2.6 (a), we have that $\sigma(\mathcal{C}) \subset \mathcal{E}$.

Let f be a $\sigma(\mathcal{C})$-measurable extended real valued function on F. Put

$$f_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} 1_{\{k/2^n \leq f < (k+1)/2^n\}}.$$

Then $f_n \in \mathcal{V},\ 0 \leq f_n \uparrow f^+$, and $f^+ \in \mathcal{V}$. By the same argument, we have $f^- \in \mathcal{V}$. Hence $f = f^+ - f^- \in \mathcal{V}$. □

Remark 2.9. Note that, to conclude that $\mathcal{V}$ contains all $\sigma(\mathcal{C})$-measurable finite (resp. bounded) functions, it is sufficient the conditions to hold for all finite (resp. bounded) functions.

Theorem 2.10. (Doob's measurablility theorem). *Let f be a mapping from E to a measurable space $(F, \mathcal{F})$, and ϕ be an extended real valued function on E. Then ϕ is $\sigma(f)$-measurable if and only if there exists an $\mathcal{F}$-measurable extended real valued function h on F such that $\phi = h \circ f$.*

Proof. The sufficiency is trivial. We shall show the necessity. Put

$$\mathcal{H} = \{h \circ f : h \text{ is a } \mathcal{F}\text{-measurable function on } F\}$$

Let $A \in \sigma(f)$. There exists $B \in \mathcal{F}$ such that $A = f^{-1}(B)$. Hence, $1_A = 1_B \circ f \in \mathcal{H}$. For any $\alpha, \beta \in \mathbb{R}$, and $f, g \in \mathcal{V}$, we have $\alpha f + \beta g \in \mathcal{V}$. Let (h_n) be $\mathcal{F}$-measurable functions on F such that $0 \leq h_n \uparrow h$. By monotone convergence theorem, h is an $\mathcal{F}$-measurable function on F. Then $h_n \circ f \in \mathcal{H}, 0 \leq h_n \circ f \uparrow h \circ f \in \mathcal{H}$. By Theorem 2.8, $\mathcal{H}$ contains all $\sigma(f)$-measurable functions. This means that for every $\sigma(f)$-measurable ϕ there exists $\mathcal{F}$-measurable function h such that $\phi = h \circ f$. □

3 Descriptive Set Theory

Let $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ be two Borel spaces, π denote the Cartesian coordinate projection mapping from the product space $X \times Y$ onto X, and $(B_n) \subset \mathcal{B}(X \times Y)$ be such that $B_n \downarrow B \neq \emptyset$. Consider the following

$$\pi(\bigcap_{n=1}^{\infty} B_n) = \bigcap_{n=1}^{\infty} \pi(B_n).$$

This is a famous mistake in the proof of the following (false) statement: *if the set is Borel measurable, then its projection is Borel measurable*; see Lebesgue [1905]. Henri Lebesgue was well aware that taking a countable intersection and taking a projection are not commutative operations for Borel sets, but thought that assuming the intersection to be non-empty he could evade this difficulty. However, this should hold everywhere to guarantee the commutativity, which breaks the argument.

The error was spotted ten years later by Mikhail Suslin. Suslin constructed a Borel set, which has a non-Borel measurable projection; see Souslin [1917]. This incident had a major influence in the early development of descriptive set theory. Remark that, if the Borel class $\mathcal{B}(X \times Y)$ is replaced with the class of compact sets $\mathcal{K}(X \times Y)$, then the operations of taking a projection and a countable intersection commute.

3.1 Analytic Sets

Let F be a set, and $\mathcal{F}$ be a paving on F. A subset A of F is called $\mathcal{F}$-*analytic*, if there exists a compact metrizable space E, and a subset B of $F \times E$ belonging to $(\mathcal{F} \times \mathcal{K}(E))_{\sigma\delta}$ such that A is the projection of B onto F. The class of $\mathcal{F}$-analytic sets is denoted by $\mathcal{A}(\mathcal{F})$. It is apparent, from the definition, that

$$\mathcal{F} \subset \mathcal{A}(\mathcal{F}). \tag{11}$$

Our motivation to study analytic sets originates from their close, however non-trivial, relationship to σ-algebras; see Proposition 3.1, Theorem 3.3 and Theorem 3.10.

Proposition 3.1. *Let $\mathcal{F}$ be a paving on F. Then*
(a) *$\mathcal{A}(\mathcal{F})$ is closed under countable unions and intersections;*
(b) *$\sigma(\mathcal{F}) \subset \mathcal{A}(\mathcal{F})$ if and only if $A^c \in \mathcal{A}(\mathcal{F})$ for every $A \in \mathcal{F}$.*

Proof. (a) Let $(A_n) \subset \mathcal{A}(\mathcal{F})$ and respectively $B_n \in (\mathcal{F} \times \mathcal{K}(E_n))_{\sigma\delta}$, where E_n is compact, be such that A_n is the projection of B_n onto F. Let E denote the Cartesian product space $\prod_n E_n$ and π denote the projection mapping from $F \times E$ onto F. Put

$$C_n = B_n \times \prod_{m \neq n} E_m.$$

We have that

$$\bigcap_n A_n = \pi\big(\prod_n B_n\big) = \pi\big(\bigcap_n C_n\big).$$

Since $B_n \in (\mathcal{F} \times \mathcal{K}(E_n))_{\sigma\delta}$, we may write $B_n = \bigcap_k B_{n,k}$, where $B_{n,k} \in (\mathcal{F} \times \mathcal{K}(E_n))_{\sigma}$ for each k. Since $B_{n,k} \times \prod_{m \neq n} E_m \in (\mathcal{F} \times \mathcal{K}(E))_{\sigma}$ for each k, we have that $C_n \in (\mathcal{F} \times \mathcal{K}(E))_{\sigma\delta}$ for each n. Hence, $\bigcap_n C_n \in (\mathcal{F} \times \mathcal{K}(E))_{\sigma\delta}$. Thus, $\bigcap_n A_n \in \mathcal{A}(\mathcal{F})$, i.e., $\mathcal{A}(\mathcal{F})$ is closed under countable intersections.

Before we proceed, let us recall that any topological space X can be compactified by adding one extra point, say the point denoted by ∞, and enlarging the topology with the sets of the form $O \cup \{\infty\}$, where O is open in X and such that $X \setminus O$ is closed and compact in X. This is called *Aleksandrov's one-point compactification*; see Alexandroff [1924].

Let A_n, B_n and E_n be as above, E denote the one-point compactification of the coproduct space: $\coprod_n E_n = \{(x,n) : x \in E_n\}$, and π denote the projection mapping from $F \times E$ onto F. We may identify $F \times \coprod_n E_n$ with $\coprod_n (F \times E_n)$. Then

$$\bigcup_n A_n = \pi(\coprod_n B_n).$$

Again, we may write $B_n = \bigcap_k B_{n,k}$, where $B_{n,k} \in (\mathcal{F} \times \mathcal{K}(E_n))_\sigma$ for each k. Since $\coprod_n B_{n,k} \in (\mathcal{F} \times \mathcal{K}(E))_\sigma$ for each k, we have

$$\coprod_n B_n = \coprod_n \bigcap_k B_{n,k} = \bigcap_k \coprod_n B_{n,k} \in (\mathcal{F} \times \mathcal{K}(E))_{\sigma\delta}.$$

Thus, $\bigcup_n A_n \in \mathcal{A}(\mathcal{F})$, i.e., $\mathcal{A}(\mathcal{F})$ is closed under countable unions.

(b) The necessity is trivial. We show the sufficiency. Put

$$\mathcal{G} = \{A \in \mathcal{A}(\mathcal{F}) : A^c \in \mathcal{A}(\mathcal{F})\}.$$

By (a), $\mathcal{G}$ is a σ-algebra. Since $\mathcal{F} \subset \mathcal{G}$, we have

$$\sigma(\mathcal{F}) \subset \mathcal{G} \subset \mathcal{A}(\mathcal{F}).$$

□

Lemma 3.2. *Let $\mathcal{F}$ be a pavings on F. Then*

(a) *for $\mathcal{F} \cap A = \{A' \cap A : A' \in \mathcal{F}\}$, $A \in \mathcal{F}$, we have*

$$\mathcal{A}(\mathcal{F}) \cap A = \mathcal{A}(\mathcal{F} \cap A);$$

(b) *for each compact metrizable space E and $B \in \mathcal{A}(\mathcal{F} \times \mathcal{K}(E))$, we have*

$$\pi_{F\times E\to F}(B) \in \mathcal{A}(\mathcal{F});$$

(c) *for each paving $\mathcal{G}$ on F such that $\mathcal{F} \subset \mathcal{G} \subset \mathcal{A}(\mathcal{F})$, we have*

$$\mathcal{A}(\mathcal{F}) = \mathcal{A}(\mathcal{G}) = \mathcal{A}(\mathcal{A}(\mathcal{F})).$$

Proof. (a) Let $B \in \mathcal{A}(\mathcal{F} \cap A)$. There exists a compact metrizable space E such that B is the projection of C onto A for some $C \in ((\mathcal{F} \cap A) \times \mathcal{K}(E))_{\sigma\delta}$, say $C = \bigcap_{n=1}^\infty \bigcup_{m=1}^\infty C_{n,m}$, where $C_{n,m} \in (\mathcal{F} \cap A) \times \mathcal{K}(E)$. From the definition of the paving $\mathcal{F} \cap A$, we have, for all n, m, $C_{n,m} = C'_{n,m} \cap (A \times E)$ for some $C'_{n,m} \in \mathcal{F} \times \mathcal{K}(E)$, and furthermore $C = C' \cap (A \times E)$ for $C' = \bigcap_{n=1}^\infty \bigcup_{m=1}^\infty C'_{n,m} \in (\mathcal{F} \times \mathcal{K}(E))_{\sigma\delta}$. Hence, we have $B = \pi(C') \cap A \in \mathcal{A}(\mathcal{F}) \cap A$, where π is the projection from $F \times E$ onto F. This means that $\mathcal{A}(\mathcal{F} \cap A) \subset \mathcal{A}(\mathcal{F}) \cap A$.

Conversely, let us assume that $D \in \mathcal{A}(\mathcal{F})$. Then $D = \pi(G)$ for some $G \in (\mathcal{F} \times \mathcal{K}(E))_{\sigma\delta}$. Since $G \cap (A \times E) \in ((\mathcal{F} \cap A) \times \mathcal{K}(E))_{\sigma\delta}$ and $D \cap A = \pi(G \cap (A \times E))$, we have $D \cap A \in \mathcal{A}(\mathcal{F} \cap A)$. This means that $\mathcal{A}(\mathcal{F}) \cap A \subset \mathcal{A}(\mathcal{F} \cap A)$, which completes the proof.

(b) Let A denote the projection of B onto F. There exists a compact metrizable space H and $C \in (\mathcal{F} \times \mathcal{K}(E) \times \mathcal{K}(H))_{\sigma\delta}$ such that B is the projection of C onto $F \times E$. But $\mathcal{K}(E) \times \mathcal{K}(H) \subset \mathcal{K}(E \times H)$, the space $E \times H$ is compact and metrizable, and A is the projection of C onto F. Thus, $A \in \mathcal{A}(\mathcal{F})$.

(c) We have that $\mathcal{A}(\mathcal{F}) \subset \mathcal{A}(\mathcal{G}) \subset \mathcal{A}(\mathcal{A}(\mathcal{F}))$, and hence, it is sufficient to show that $\mathcal{A}(\mathcal{F}) \supset \mathcal{A}(\mathcal{A}(\mathcal{F}))$. Let $A \in \mathcal{A}(\mathcal{A}(\mathcal{F}))$. By the definition, there exists a compact metrizable space E and $B \in (\mathcal{A}(\mathcal{F}) \times \mathcal{K}(E))_{\sigma\delta}$ such that A is the projection of B onto F. We have

$$\mathcal{A}(\mathcal{F}) \times \mathcal{K}(E) \subset \mathcal{A}(\mathcal{F} \times \mathcal{K}(E)).$$

Hence, by Proposition 3.1 (a), $B \in \mathcal{A}(\mathcal{F} \times \mathcal{K}(E))$, and, by the previous assertion (b), $A \in \mathcal{A}(\mathcal{F})$. This completes the proof. □

Theorem 3.3. *Let $(F, \mathcal{F})$ be a measurable space, $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$, $\mathcal{K} = \mathcal{K}(\mathbb{R}^d)$, $\mathcal{F} \otimes \mathcal{B}$ be the product σ-algebra on $F \times \mathbb{R}^d$.*

(a) *$\mathcal{F} \otimes \mathcal{B} \subset \mathcal{A}(\mathcal{F} \times \mathcal{K}) = \mathcal{A}(\mathcal{F} \otimes \mathcal{B})$;*

(b) *For any $A \in \mathcal{A}(\mathcal{F} \otimes \mathcal{B})$, the projection of A onto F is $\mathcal{F}$-analytic.*

Proof. (a) Let $B \in \mathcal{F} \times \mathcal{K}$. We have that $B^c \in (\mathcal{F} \times \mathcal{K})_\sigma \subset \mathcal{A}(\mathcal{F} \times \mathcal{K})$. Since $\mathcal{F} \otimes \mathcal{B} = \sigma(\mathcal{F} \times \mathcal{K})$, we have $\mathcal{F} \times \mathcal{K} \subset \mathcal{F} \otimes \mathcal{B} \subset \mathcal{A}(\mathcal{F} \times \mathcal{K})$ by Proposition 3.1 (b), and $\mathcal{A}(\mathcal{F} \times \mathcal{K}) = \mathcal{A}(\mathcal{F} \otimes \mathcal{B})$ by Lemma 3.2 (c).

(b) Take $(K_n) \subset \mathcal{K}$ such that $\bigcup_n K_n = \mathbb{R}^d$. By Lemma 3.2 (a), for each n we have

$$\begin{aligned}\mathcal{A}(\mathcal{F} \times \mathcal{K}) \cap (F \times K_n) &= \mathcal{A}((\mathcal{F} \times \mathcal{K}) \cap (F \times K_n)) \\ &= \mathcal{A}(\mathcal{F} \times (\mathcal{K} \cap K_n)) \\ &= \mathcal{A}(\mathcal{F} \times \mathcal{K}(K_n)).\end{aligned}$$

From the assertion (a) we have that $A \in \mathcal{A}(\mathcal{F} \times \mathcal{K})$, so, by Lemma 3.2 (b), the projection of $A \cap (F \times K_n)$ onto F is $\mathcal{F}$-analytic for every n. We write $A = \bigcup_n (A \cap (F \times K_n))$ and conclude that the projection of A onto F is $\mathcal{F}$-analytic. □

Remark 3.4. One can replace $\mathbb{R}^d$ with $\mathbb{R}_+$ in Theorem 3.3.

3.2 Capacity

Let $(F, \mathcal{F})$ be a paved set, where $\mathcal{F}$ is closed under formation of countable union and intersection. An extended real valued mapping I from the subsets of F, defined for every subset of F, is called an *$\mathcal{F}$-capacity* on F, if

(a) I is increasing: $A \subset B \implies I(A) \leq I(B)$;

(b) I is continuous from below: $A_n \uparrow A \implies I(A_n) \uparrow I(A)$;

(c) I is continuous in $\mathcal{F}$ from above: $A_n \in \mathcal{F}, A_n \downarrow A \implies I(A_n) \downarrow I(A)$.

The capacity was introduced by Gustave Choquet who used it to approximate the Borel sets; see Choquet [1954].

Example 3.5. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Then

$$\overline{\mathbf{P}} = \inf\{\mathbf{P}(B) : B \in \mathcal{F}, A \subset B\},\ A \subset \Omega,$$

is an $\mathcal{F}$-capacity on Ω.

Let I be an $\mathcal{F}$-capacity on F. A set $A \subset F$ is called I-*capacitable*, if

$$I(A) = \sup\{I(B) : B \in \mathcal{F}_\delta, B \subset A\}.$$

Lemma 3.6. *Let I be an $\mathcal{F}$-capacity on F. Then each $A \in \mathcal{F}_{\sigma\delta}$ is I-capacitable.*

Proof. If $I(A) = -\infty$, then $I(\emptyset) = -\infty$, i.e., A is I-capacitable. Assume $I(A) > -\infty$. Since A is an element of $\mathcal{F}_{\sigma\delta}$, we may write

$$A = \bigcap_{n=1}^{\infty} A_n,\ A_n \in \mathcal{F}_\sigma.$$

$$A_n = \bigcup_{n=1}^{\infty} A_{n,m},\ A_{n,m} \in \mathcal{F}.$$

Since $\mathcal{F}$ is closed under taking the finite union, we may assume that, for every $n \geq 1$, the sequence $(A_{m,n})_m$, $m \geq 1$, is increasing. To show that the set A is I-capacitable, it is sufficient to show that for every $a < I(A)$ there exists $B \subset A$, $B \in \mathcal{F}_\delta$, such that $I(B) \geq a$.

Assume $a < I(A)$. Since I is continuous from below, we have

$$I(A) = I(A \cap A_1) = \sup_m I(A \cap A_{1,m}).$$

Hence, there exists an integer m_1 such that $I(A \cap A_{1,m_1}) > a$. Then, by induction, there exists a sequence $(m_k) \subset \mathbb{N}$ such that for every $k \geq 1$ we have

$$I(A \cap A_{1,m_1} \cap \cdots \cap A_{k,m_k}) > a.$$

Put $B_n = \bigcap_{k=1}^n A_{k,m_k}$. By monotonicity of I, we have that $I(B_n) > a$. On the other hand, since $B_n \in \mathcal{F}$, $B_n \downarrow B \in \mathcal{F}_\delta$ and I is continuous in $\mathcal{F}$ from above, we have that $I(B) = \inf_n I(B_n) \geq a$. From $B_n \subset A_n$, we get $B \subset A$. This completes the proof. $\square$

Lemma 3.7. *Let I be an $\mathcal{F}$-capacity on F, E be a compact metrizable space and π denote the projection mapping from $F \times E$ onto F. Then*

$$J(G) = I(\pi(G)),\ G \subset F \times E,$$

is a $\mathcal{G}$-capacity on $F \times E$, where $\mathcal{G} = \{\bigcup_{k=1}^n G_k,\ G_k \in \mathcal{F} \times \mathcal{K}(E)\}$. In addition, we have $\pi(A) \in \mathcal{F}_\delta$ for every $A \in \mathcal{G}_\delta$.

Proof. It is easily seen that the paving $\mathcal{G}$ is closed under formation of finite union and intersection. Properties (a) and (b) in the definition of capacity are satisfied by J. Let us verify property (c).

Let $G = \bigcup_{k=1}^{m} F_k \times K_k$, where $F_k \in \mathcal{F}$ and $K_k \in \mathcal{K}(E)$. For each $x \in \pi(G)$, we have

$$(\{x \times E\}) \cap G = \{x\} \times K,$$

where the set $K = \bigcup_{\{k:x\in F_k\}} K_k \in \mathcal{K}(E)$ is non-empty. Now let $A_n \in \mathcal{G}$, $A_n \downarrow A \in \mathcal{G}_\delta$ and $x \in \bigcap_{n=1}^{\infty} \pi(A_n)$. For each n there exists a non-empty $K_n \in \mathcal{K}(E)$ such that

$$(\{x \times E\}) \cap A_n = \{x\} \times K_n.$$

Since (A_n) is decreasing, so is (K_n). Moreover, since each K_n is non-empty and compact, we have $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$, and

$$\big(\{x \times E\}\big) \cap \big(\bigcap_{n=1}^{\infty} A_n\big) = \{x\} \times \bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

Hence, $x \in \pi\big(\bigcap_{n=1}^{\infty} A_n\big)$, i.e., $\bigcap_{n=1}^{\infty} \pi(A_n) \subset \pi\big(\bigcap_{n=1}^{\infty} A_n\big)$. Since the reverse inclusion always holds, we have

$$\bigcap_{n=1}^{\infty} \pi(A_n) = \pi\big(\bigcap_{n=1}^{\infty} A_n\big).$$

This means that $\pi(A) \in \mathcal{F}_\delta$ for every $A \in \mathcal{G}_\delta$. Furthermore, since $\pi(A_n) \in \mathcal{F}$, $\pi(A_n) \downarrow \pi(A) \in \mathcal{F}_\delta$, we have that

$$J\big(\bigcap_n A_n\big) = I\big(\pi\big(\bigcap_n A_n\big)\big) = I\big(\bigcap_n \pi(A_n)\big) = \inf_n I(\pi(A_n)) = \inf_n J(A_n),$$

i.e., property (c) in holds for J. Hence, J is a $\mathcal{G}$-capacity on $F \times E$. □

Theorem 3.8. (Choquet's Theorem). *Let I be an $\mathcal{F}$-capacity on F. Then each $A \in \mathcal{A}(\mathcal{F})$ is I-capacitable.*

Proof. Let $A \in \mathcal{A}(\mathcal{F})$. Then there exists a compact metrizable space E, and $B \in (\mathcal{F} \times \mathcal{K}(E))_{\sigma\delta}$ such that $A = \pi(B)$, where π is the projection mapping from $F \times E$ onto F. Let $\mathcal{G} = \{\bigcup_{k=1}^{n} G_k,\ G_k \in \mathcal{F} \times \mathcal{K}(E)\}$ and $J(G) = I(\pi(G)),\ G \subset F \times E$. Then, by Lemma 3.7, J is a $\mathcal{G}$-capacity on $F \times E$. We see that $\mathcal{G}_{\sigma\delta} = (\mathcal{F} \times \mathcal{K}(E))_{\sigma\delta}$. Hence, $B \in \mathcal{G}_{\sigma\delta}$, and by Lemma 3.6, B is J-capacitable. On the other hand, by Lemma 3.7, if $C \in \mathcal{G}_\delta$, then $\pi(C) \in \mathcal{F}_\delta$. Hence

$$I(A) = J(B) = \sup_{C\in\mathcal{G}_\delta, C\subset B} J(C) = \sup_{C\in\mathcal{G}_\delta, C\subset B} I(\pi(C)) \le \sup_{D\in\mathcal{F}_\delta, D\subset A} I(D).$$

But since $I(A) \ge \sup_{D\in\mathcal{F}_\delta, D\subset A} I(D)$, we have

$$I(A) = \sup_{D\in\mathcal{F}_\delta, D\subset A} I(D).$$

This means that A is I-capacitable. □

3.3 Universally Measurable Sets

Let $(F, \mathcal{F})$ be a measurable space and μ a σ-finite measure on $(F, \mathcal{F})$. A set $A \subset F$ is a *μ-null set* if $A \subset B \in \mathcal{F}$ with $\mu(B) = 0$. A σ-algebra is said to be *complete* for μ, if it contains all μ-null sets. The smallest σ-algebra which contains $\mathcal{F}$ and is complete for μ is called the *μ-completion* of $\mathcal{F}$ and denoted with $\mathcal{F}^\mu$. The σ-algebra

$$\widehat{\mathcal{F}} = \bigcap_{\mu} \mathcal{F}^\mu,$$

where μ ranges over all σ-finite measures on $(F, \mathcal{F})$, is called the *universal completion* of $\mathcal{F}$. The elements in $\widehat{\mathcal{F}}$ are called *universally measurable sets*.

Remark 3.9. For every μ, we have $\mathcal{F} \subset \widehat{\mathcal{F}} \subset \mathcal{F}^\mu$. If $\mathcal{F} = \mathcal{F}^\mu$ for some μ, then $\mathcal{F} = \widehat{\mathcal{F}} = \mathcal{F}^\mu$.

Theorem 3.10. *Let $(F, \mathcal{F})$ be a measurable space. Then*

$$\mathcal{A}(\mathcal{F}) \subset \widehat{\mathcal{F}}.$$

Proof. Let μ be a σ-finite measure on $(F, \mathcal{F})$. Then

$$I(A) = \inf\{\mu(B) : B \in \mathcal{F}, A \subset B\}, \ A \subset F,$$

is an $\mathcal{F}$-capacity on F. We have $\mathcal{F}_\delta = \mathcal{F}$. Hence, by Theorem 3.8, for each $A \in \mathcal{A}(\mathcal{F})$ we have

$$I(A) = \sup\{\mu(B) : B \in \mathcal{F}, B \subset A\}.$$

So, for every $A \in \mathcal{A}(\mathcal{F})$, there exists $B, C \in \mathcal{F}$ such that $B \subset A \subset C$ and $\mu(B) = \mu(C)$. This means that $A \in \mathcal{F}^\mu$. Because μ is arbitrary, $A \in \widehat{\mathcal{F}}$. $\square$

Remark 3.11. By (11) and Theorem 3.10, we have $\mathcal{F} \subset \mathcal{A}(\mathcal{F}) \subset \widehat{\mathcal{F}}$. If $\mathcal{F} = \widehat{\mathcal{F}}$, then $\mathcal{F} = \mathcal{A}(\mathcal{F}) = \widehat{\mathcal{F}}$.

Let $(\Omega, \mathcal{F})$ be a measurable space, and $\Gamma : \Omega \rightrightarrows \mathbb{R}^d$ be a set-valued mapping. Recall the notions of domain $\mathrm{dom}(\Gamma) = \{\omega \in \Omega : \Gamma(\omega) \neq \emptyset\}$ and graph $\mathrm{Gr}(\Gamma) = \{(\omega, x) : x \in \Gamma(\omega)\}$. We say that a mapping $v : \Omega \to \mathbb{R}^d$ is a *section* of $\mathrm{Gr}(\Gamma)$, if

$$(\omega, v(\omega)) \subset \mathrm{Gr}(\Gamma) \text{ for all } \omega \in \mathrm{dom}(\Gamma). \tag{12}$$

In the literature, one often sees the condition (12) written in the equal form;

$$v(\omega) \in \Gamma(\omega) \text{ for all } \omega \in \mathrm{dom}(\Gamma),$$

and then v called a *selection* of Γ. The terms *cross section* and *selector* are used as well.

Theorem 3.12. *Let $(\Omega, \mathcal{F})$ be a measurable space, and $\Gamma : \Omega \rightrightarrows \mathbb{R}^d$ be a set-valued mapping such that $\mathrm{Gr}(\Gamma) \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$. Then there exists an $\widehat{\mathcal{F}}$-measurable section of $\mathrm{Gr}(\Gamma)$.*

Proof. See e.g. Aumann [1967]. $\square$

The sections of $\Omega \times \mathbb{R}_+$ are treated in more detail in Section 4.2.

4 Stochastic Processes

In Section 4.1 we introduce two important sub-σ-algebras of $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$, the one generated by adapted right-continuous processes and the one generated by adapted left-continuous processes, namely the optional and the predictable σ-algebra. A particular attention is paid on their relation with the stopping times.

Let A be a subset of $\Omega \times \mathbb{R}_+$. Then, by the axiom of choice, there exists a mapping τ such that $(\omega, \tau(\omega)) \in A$. The interesting question is that if the set A is assumed to be measurable, can τ be chosen to be measurable as well? The answer to this are so-called *section theorems*. In Section 4.2 we prove the optional and the predictable section theorem.

Section 4.3 is devoted for classical martingale theory. We prove *Föllmer's lemma* and optional and predictable version of *Doob's stopping theorem*. In Section 4.4 we use these results and the section theorems of Section 4.2 to prove the existence and uniqueness of optional and predictable projection of a measurable process.

4.1 Stopping Times

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, I be an well ordered subset of $\overline{\mathbb{R}}_+$, and $(\mathcal{F}_i)_{i\in I}$ a filtration of $\mathcal{F}$. An I-valued random variable is called an $(\mathcal{F}_i)_{i\in I}$-*stopping time*, if

$$\{\tau \leq i\} \in \mathcal{F}_i \text{ for each } i \in I.$$

Note that, when I is denumerable, $\{\tau \leq i\} \in \mathcal{F}_i$ if and only if $\{\tau = i\} \in \mathcal{F}_i$.

Put $\mathcal{F}_\infty = \bigvee_{i\in I} \mathcal{F}_i$ and

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq i\} \in \mathcal{F}_i \ \forall i \in I\}.$$

We call $\mathcal{F}_\tau$ the *σ-algebra of events prior to τ*.

Put $\mathcal{F}_0 = \bigwedge_{i\in I} \mathcal{F}_i$ and

$$\mathcal{F}_{\tau-} = \mathcal{F}_0 \vee \sigma\{A \cap \{i < \tau\} : A \in \mathcal{F}_i,\ i \in I\}.$$

We call $\mathcal{F}_{\tau-}$ the *σ-algebra of events strictly prior to τ*. We have $\mathcal{F}_{\tau-} \subset \mathcal{F}_\tau$.

Let τ be an I-valued function on Ω and $A \subset \Omega$. Put

$$\tau_A = \tau 1_A + \infty 1_{A^c}.$$

We call τ_A the *restriction of τ on A*.

Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration of $\mathcal{F}$ satisfying the *usual conditions*, i.e., $\mathcal{F}_0$ contains $\mathbf{P}$-null sets and $\mathbb{F}$ is right-continuous: $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for each $t \geq 0$. Henceforth, if nothing dictates otherwise, $I = \overline{\mathbb{R}}_+$ and all stopping times are relative to $\mathbb{F}$. We will denote the family of all stopping times by $\mathcal{T}$.

Proposition 4.1. *We have*

(a) $\tau \equiv s \in \overline{\mathbb{R}}_+ \implies \tau \in \mathcal{T}$,

(b) $(\tau_k) \subset \mathcal{T}, k \in \mathbb{N} \implies \bigwedge_{k=1}^n \tau_k \in \mathcal{T},\ \bigvee_{k=1}^\infty \tau_k \in \mathcal{T}$,

(c) $\sigma, \tau \in \mathcal{T} \implies \sigma_{\{\sigma<\tau\}} \in \mathcal{T}$.

Proof. (a) Clearly, $\{\tau \leq t\} = \Omega$ if $s \leq t$ and $\{\tau \leq t\} = \emptyset$ if $s > t$. Hence, a constant is a stopping time.

(b) We have $\{\bigwedge_{k=1}^{n} \tau_k > t\} = \bigcap_{k=1}^{n}\{\tau_k > t\}$ and $\{\bigvee_{k=1}^{\infty} \tau_k \leq t\} = \bigcap_{k=1}^{\infty}\{\tau_k \leq t\}$. Hence, $\bigwedge_{k=1}^{n} \tau_k$ and $\bigvee_{k=1}^{\infty} \tau_k$ are stopping times.

(c) We have $\{\sigma < \tau\}^c = \{\tau \leq \sigma\}$, and by (a) and (b) $\{\tau \wedge t \leq s\} \in \mathcal{F}_s$ for all $0 \leq s \leq t$. Thereby,

$$\{\sigma < \tau\}^c \cap \{\sigma \leq t\} = \{\tau \leq \sigma\} \cap \{\sigma \leq t\} = \{\tau \leq t\} \cap \{\sigma \leq t\} \cap \{\tau \wedge t \leq \sigma \wedge t\} \in \mathcal{F}_t,$$

for each $t \geq 0$. Hence, $\{\sigma < \tau\} \in \mathcal{F}_\sigma$, and as a consequence $\sigma_{\{\sigma<\tau\}}$ is $\mathcal{F}_\sigma$-measurable. For each $t \geq 0$, we have $\{\sigma_{\{\sigma<\tau\}} \leq t\} \in \mathcal{F}_\sigma$, and $\sigma \leq \sigma_{\{\sigma<\tau\}}$. Thus,

$$\{\sigma_{\{\sigma<\tau\}} \leq t\} = \{\sigma_{\{\sigma<\tau\}} \leq t\} \cap \{\sigma \leq t\} \in \mathcal{F}_t,$$

for each $t \geq 0$. This means that $\sigma_{\{\sigma<\tau\}}$ is a stopping time. □

Remark 4.2. It is easy to see that in general $\bigwedge_{k=1}^{\infty} \tau_k$ is not a stopping time, but if (τ_k), $k \in \mathbb{N}$, is *stationary*, i.e., such that for every $\omega \in \Omega$ there exists $n \in \mathbb{N}$ such that $\tau_k(\omega) = \tau_n(\omega)$ for $k \geq n$. Then $\{\bigwedge_{k=1}^{\infty} \tau_k > t\} = \bigcap_{k=1}^{\infty}\{\tau_k > t\} \in \mathcal{F}_t$ for $t \geq 0$, and $\bigwedge_{k=1}^{\infty} \tau_k$ is a stopping time.

Let I be a well ordered subset of $\overline{\mathbb{R}}$, and $(\mathcal{F}_i)_{i\in I}$ a filtration of $\mathcal{F}$. A *stochastic process* v is a collection $(v_i)_{i\in I}$, where each v_i is a real valued random variable on Ω. We define an $\mathbb{R}^d$-valued process be such that its every component is a real valued process, and so, all the relevant discussion holds for the $\mathbb{R}^d$ valued processes as well. If every v_i is $\mathcal{F}_i$-measurable, we say that the process v is *adapted* to the filtration $(\mathcal{F}_i)_{i\in I}$. In this chapter we assume $I = \mathbb{R}_+$. Let $v = (v_t)$, $t \in \mathbb{R}_+$, be a stochastic process. If $v_t(\omega)$, as a function of (ω, t), is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable, we say that the process v is *measurable*.

A subset $A \subset \Omega \times \mathbb{R}_+$ is called a *stochastic set*, if its indicator is a stochastic process: $1_A = ((1_A)_t)_{t\geq 0}$, where $(1_A)_t = 1_{A_t}$ and A_t is the section of A at t; $A_t = \{(\omega, t) : t \in A\}$. A stochastic set A is said to be measurable, if the corresponding indicator process 1_A is measurable, i.e., $A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$.

Let σ and τ be two $\overline{\mathbb{R}}_+$-valued random variables. Then the *stochastic interval* $[\sigma, \tau] = \{(\omega, t) : \sigma(\omega) \leq t \leq \tau(\omega)\}$ is a stochastic set. Open and half-open stochastic intervals are defined in a similar manner.

The σ-algebra on $\Omega \times \mathbb{R}_+$ generated by all adapted right-continuous processes is called the *optional σ-algebra* and denoted by $\mathcal{O}$. Every adapted right-continuous process can be written as a limit of $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable simple processes, and therefore $\mathcal{O} \subset \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$. The σ-algebra generated by all adapted left-continuous processes is called the *predictable σ-algebra* and denoted by $\mathcal{P}$. One can show that $\mathcal{P} \subset \mathcal{O}$ (cf. Remark 4.14). A stochastic set or process is called *optional*, if it is $\mathcal{O}$-measurable, and *predictable*, if it is $\mathcal{P}$-measurable.

The predictable processes are particularly important, since they play a central role in the famous *Doob-Meyer decomposition* of semimartingales, and are used extensively as integrands of stochastic integrals in stochastic calculus; see details from Dellacherie and Meyer [1982] or He Sheng-wu and Jia-an [1992]. The intuitive idea of a predictable process is that, if the filtration is interpreted as an increasing information about the values of the process, then, for a predictable process, every value is observable a differential amount of time in advance.

Proposition 4.3. *Put*

$$\mathcal{C} = \{[\tau, \infty[: \tau \in \mathcal{T}\}.$$

Then $\sigma(\mathcal{C}) = \mathcal{O}$.

Proof. Since $1_{[\tau,\infty[}$ is adapted and right-continuous, $\mathcal{C} \subset \mathcal{O}$. It suffices to show that $\mathcal{O} \subset \sigma(\mathcal{C})$, i.e., if $v = (v_t)$ is an adapted right-continuous process, then v is $\sigma(\mathcal{C})$-measurable. We will show that for any given $\epsilon > 0$ there exists a $\sigma(\mathcal{C})$-measurable process v^ϵ such that $|v^\epsilon_t(\omega) - v_t(\omega)| < \epsilon$ for all $(\omega, t) \in \Omega \times \mathbb{R}_+$.

Let us first show that for any given $\epsilon > 0$ there exists a sequence $(\tau^\epsilon_n) \subset \mathcal{T}$, $\tau^\epsilon_n \uparrow \infty$, such that $\tau^\epsilon_{n+1}(\omega) > \tau^\epsilon_n(\omega)$ when $\tau^\epsilon_{n+1}(\omega) < \infty$, and

$$|v_{\tau^\epsilon_n} - v_t| < \epsilon \text{ for all } t \in [\tau^\epsilon_n, \tau^\epsilon_{n+1}[. \tag{13}$$

Put $\tau^\epsilon_0 = 0$, and

$$\tau^\epsilon_{n+1} = \inf\{t : t > \tau^\epsilon_n, |v_{\tau^\epsilon_n} - v_t| \geq \epsilon\} \wedge \inf\{t : t > \tau^\epsilon_n, |v_{\tau^\epsilon_n} - v_{t-}| \geq \epsilon\}.$$

It is clear that the sequence (τ^ϵ_n) is strictly increasing when the values are finite, tends to infinity, and satisfies (13). It remains to verify that τ^ϵ_n are stopping times. We proceed by induction. Assume τ^ϵ_n is a stopping time. We have

$$\{\tau^\epsilon_{n+1} = r\} \subset \{\tau^\epsilon_n < r\} \cap \big(\{|v_{\tau^\epsilon_n} - v_r| \geq \epsilon\} \cup \{|v_{\tau^\epsilon_n} - v_{r-}| \geq \epsilon\}\big) \subset \{\tau^\epsilon_{n+1} \leq r\}.$$

for every $r \in \mathbb{R}_+$, and

$$\{\tau^\epsilon_{n+1} \leq t\} = \bigcup_{r \leq t} \{\tau^\epsilon_{n+1} = r\} = \bigcup_{r \leq t} \{\tau^\epsilon_{n+1} \leq r\}.$$

We obtain

$$\begin{aligned}\{\tau^\epsilon_{n+1} \leq t\} &= \bigcup_{r \leq t} \big(\{\tau^\epsilon_n < r\} \cap \big(\{|v_{\tau^\epsilon_n} - v_r| \geq \epsilon\} \cup \{|v_{\tau^\epsilon_n} - v_{r-}| \geq \epsilon\}\big)\big) \\ &= \bigcap_{m=1}^{\infty} \bigcup_{q \in \mathbb{Q}_t} \big(\{\tau^\epsilon_n < q\} \cap \{|v_{\tau^\epsilon_n} - v_q| > \epsilon(1 - \frac{1}{m})\}\big),\end{aligned}$$

where $\mathbb{Q}_t = (\mathbb{Q} \cap [0, t]) \cup \{t\}$. We have

$$\{|v_{\tau^\epsilon_n} - v_q| > \epsilon(1 - \frac{1}{m})\} \in \mathcal{F}_{\tau^\epsilon_n \vee q},$$

and therefore

$$\{\tau^\epsilon_n < q\} \cap \{|v_{\tau^\epsilon_n} - v_q| > \epsilon(1 - \frac{1}{m})\} \in \mathcal{F}_q.$$

Thus, $\{\tau^\epsilon_{n+1} \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$, i.e., τ^ϵ_{n+1} is a stopping time.

Put

$$v^\epsilon = \sum_{n=0}^{\infty} v_{\tau^\epsilon_n} 1_{[\tau^\epsilon_n, \tau^\epsilon_{n+1}[}.$$

Then v^ϵ is $\sigma(\mathcal{C})$-measurable, and so is $v = \lim_{\epsilon \to 0} v^\epsilon$. $\square$

Proposition 4.4. *Put*

$$\mathcal{C}_1 = \{A \times \{0\} : A \in \mathcal{F}_0\} \cup \{A \times]s,t] : 0 < s < t,\ s,t, \in \mathbb{Q}_+,\ A \in \bigcup_{r<s} \mathcal{F}_r\},$$
$$\mathcal{C}_2 = \{A \times \{0\} : A \in \mathcal{F}_0\} \cup \{A \times [s,t[: 0 < s < t,\ s,t, \in \mathbb{Q}_+,\ A \in \bigcup_{r<s} \mathcal{F}_r\},$$
$$\mathcal{C}_3 = \{A \times \{0\} : A \in \mathcal{F}_0\} \cup \{]\tau,\infty[: \tau \in \mathcal{T}\}.$$

Then $\sigma(\mathcal{C}_1) = \sigma(\mathcal{C}_2) = \sigma(\mathcal{C}_3) = \mathcal{P}$.

Proof. We have $\mathcal{C}_1 \subset \mathcal{P}$. It suffices to show that $\mathcal{P} \subset \sigma(\mathcal{C}_1)$. Let $v = (v_t)$ be adapted and left-continuous. Define

$$v_t^{(n)} = v_0 1_{\{t=0\}} + \sum_{k=1}^{\infty} v_{k/2^n} 1_{\{k/2^n < t \le (k+1)/2^n\}}.$$

Then $v^{(n)}$ is $\sigma(\mathcal{C}_1)$-measurable, and so is $v = \lim_{n\to\infty} v^{(n)}$. Hence, $\mathcal{P} \subset \sigma(\mathcal{C}_1)$.

Let $A \in \mathcal{F}_r$, $r < s$. We have

$$A \times]s,t] = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} A \times [s + \frac{t-s}{n}, t + \frac{1}{m}[,$$
$$A \times [s,t[= \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} A \times]r + (1 - \frac{1}{n})(s-r), t - \frac{t-s}{n}].$$

Hence, $\mathcal{C}_1 \subset \sigma(\mathcal{C}_2)$, and $\mathcal{C}_2 \subset \sigma(\mathcal{C}_1)$. Therefore, $\sigma(\mathcal{C}_2) = \mathcal{P}$.

Since, $A \times]s,t] =]s_A, t_A]$, we have $\mathcal{C}_1 \subset \sigma(\mathcal{C}_3)$. On the other hand, $1_{]\tau,\infty]}$ is left-continuous and adapted, $\sigma(\mathcal{C}_3) \subset \mathcal{P}$, and therefore $\sigma(\mathcal{C}_3) = \mathcal{P}$. □

A stopping time τ is *predictable*, if the stochastic interval $[\tau,\infty[$ is predictable. Note that, for any stopping time τ, we have $]\tau,\infty[\in \mathcal{P}$. Since $\mathrm{Gr}(\tau) = [\tau,\infty[\setminus]\tau,\infty[$, the criteria for τ being predictable can be equivalently be written as $\mathrm{Gr}(\tau) \in \mathcal{P}$. We will denote the family of all predictable stopping times by $\mathcal{T}_p$.

Theorem 4.5. *Let τ be a stopping time. Define*

$$f(\omega) = (\omega, \tau(\omega)) \text{ on } \{\tau < \infty\}.$$

Then

$$\text{(a) } f^{-1}(\mathcal{O}) = \mathcal{F}_\tau \cap \{\tau < \infty\},$$
$$\text{(b) } f^{-1}(\mathcal{P}) = \mathcal{F}_{\tau-} \cap \{\tau < \infty\}.$$

Proof. (a) Let $\sigma \in \mathcal{T}$. We have $\{\sigma \le \tau\} \cap \{\tau \le t\} = \{\sigma \le t\} \cap \{\tau \le t\} \cap \{\sigma \wedge t \le \tau \wedge t\} \in \mathcal{F}_t$ for all $t \ge 0$. Hence, $\{\sigma \le \tau\} \in \mathcal{F}_\tau$, and $f^{-1}([\sigma,\infty[) = \{\sigma \le \tau\} \cap \{\tau < \infty\} \in \mathcal{F}_\tau \cap \{\tau < \infty\}$. Thus, $f^{-1}(\mathcal{O}) \subset \mathcal{F}_\tau \cap \{\tau < \infty\}$. Conversely, let $A \in \mathcal{F}_\infty$, such that $A \cap \{\tau \le t\} \in \mathcal{F}_t$ for all $t \ge 0$. Then

$$A \cap \{\tau < \infty\} = f^{-1}(A \times [0,\infty[) \in f^{-1}(\mathcal{O}).$$

Thus, $\mathcal{F}_\tau \cap \{\tau < \infty\} \subset f^{-1}(\mathcal{O})$.

(b) Let $A \in \mathcal{F}_0$. Then $f^{-1}(A \times \{0\}) = A \cap \{\tau = 0\} \in \mathcal{F}_{\tau-} \cap \{\tau < \infty\}$. Again, let $\sigma \in \mathcal{T}$. Then

$$\begin{aligned} f^{-1}(]\sigma, \infty[) &= \{\sigma < \tau\} \cap \{\tau < \infty\} = \bigcup_{q \in \mathbb{Q}_+} \big(\{\sigma < q\} \cap \{q < \tau\}\big) \cap \{\tau < \infty\} \\ &= \bigcup_{q \in \mathbb{Q}_+} \Big(\big(\bigcup_{n \in \mathbb{N}} \{\sigma \leq q - 1/n\}\big) \cap \{q < \tau\}\Big) \cap \{\tau < \infty\} \in \mathcal{F}_{\tau-} \cap \{\tau < \infty\}. \end{aligned}$$

Thus, $f^{-1}(\mathcal{P}) \subset \mathcal{F}_{\tau-} \cap \{\tau < \infty\}$. Conversely, let $A \in \mathcal{F}_0$. Then $A \cap \{\tau < \infty\} = f^{-1}(A \times \mathbb{R}_+) \in f^{-1}(\mathcal{P})$. Let $A \in \mathcal{F}_t$, $t \geq 0$. Then

$$(A \cap \{t < \tau\}) \cap \{\tau < \infty\} = f^{-1}(A \times]t, \infty[) \in f^{-1}(\mathcal{P}).$$

Thus, $\mathcal{F}_{\tau-} \cap \{\tau < \infty\} \subset f^{-1}(\mathcal{P})$. □

Corollary 4.6. *Let τ be a stopping time.*

(a) *Then for any optional process $v = (v_t)_{t\geq 0}$, $v_\tau 1_{\{\tau<\infty\}}$ is $\mathcal{F}_\tau$-measurable. Conversely, if x is an $\mathcal{F}_\tau$-measurable random variable, then there exists an optional process $v = (v_t)_{t\geq 0}$ such that $x1_{\{\tau<\infty\}} = v_\tau 1_{\{\tau<\infty\}}$.*

(b) *Then for any predictable process $v = (v_t)_{t\geq 0}$, $v_\tau 1_{\{\tau<\infty\}}$ is $\mathcal{F}_{\tau-}$-measurable. Conversely, if x is an $\mathcal{F}_{\tau-}$-measurable random variable, then there exists a predictable process $v = (v_t)_{t\geq 0}$ such that $x1_{\{\tau<\infty\}} = v_\tau 1_{\{\tau<\infty\}}$.*

Proof. (a) Let $f(\omega) = (\omega, \tau(\omega))$ on $\{\tau < \infty\}$ and $v = (v_t)$ be an optional process. Then, by Theorem 2.10 and Theorem 4.5 (a), $v_\tau 1_{\tau<\infty} = v \circ f$ restricted on $\{\tau < \infty\}$ is $\mathcal{F}_\tau$-measurable. Conversely, let x be an $\mathcal{F}_\tau$-measurable random variable on $\{\tau < \infty\}$. Then, by Theorem 2.10 and Theorem 4.5 (a), there exists an optional process $v = (v_t)_{t\geq 0}$ such that $x1_{\{\tau<\infty\}} = v_\tau 1_{\{\tau<\infty\}}$.

(b) The same argument applies and the assertion follow by Theorem 2.10 and Theorem 4.5 (b). □

Remark 4.7. It is easy to see from Corollary 4.6 that the optional and predictable processes are adapted.

Proposition 4.8. *We have*

(a) $\tau \equiv s \in \overline{\mathbb{R}}_+ \implies \tau \in \mathcal{T}_p$,

(b) $(\tau_k) \subset \mathcal{T}_p, k \in \mathbb{N} \implies \bigwedge_{k=1}^n \tau_k \in \mathcal{T}_p,\ \bigvee_{k=1}^\infty \tau_k \in \mathcal{T}_p$,

(c) $\sigma, \tau \in \mathcal{T}_p \implies \sigma_{\{\sigma<\tau\}} \in \mathcal{T}_p$.

Proof. By Proposition 4.1, all of the times in (a), (b) and (c) are stopping times.

(a) is evident.

(b) We have $[\bigwedge_{k=1}^n \tau_k, \infty[= \bigcup_{k=1}^n [\tau_k, \infty[$ and $[\bigvee_{k=1}^\infty \tau_k, \infty[= \bigcap_{k=1}^\infty [\tau_k, \infty[$. Hence, $\bigwedge_{k=1}^n \tau_k$ and $\bigvee_{k=1}^\infty \tau_k$ are predictable.

(c) Let us first show that $\{\sigma < \tau\} \in \mathcal{F}_{\sigma-}$. We have $\{\sigma < \tau\} = \{\tau \leq \sigma\}^c$ and $\{\tau \leq \sigma\} = \{\tau \vee \sigma = \sigma\}$. Since,

$$\{\tau \vee \sigma = \sigma\} \cap \{\sigma = \infty\} = \{\sigma = \infty\} = \bigcap_{k=1}^\infty \{k < \sigma\} \in \mathcal{F}_{\sigma-},$$

it is sufficient to show that $\{\tau \vee \sigma = \sigma\} \cap \{\sigma < \infty\} \in \mathcal{F}_{\sigma-}$. Write $1_{\{\tau\vee\sigma=\sigma\}}1_{\{\sigma<\infty\}} = 1_{\{\tau\vee\sigma<\infty\}}$. By Proposition 4.1 (b), $\tau \vee \sigma$ is a stopping time, and by Corollary 4.6 (b), $\{\tau \vee \sigma = \sigma\} \cap \{\sigma < \infty\} \in \mathcal{F}_{\sigma-}$. Thus, $\{\sigma < \tau\} \in \mathcal{F}_{\sigma-}$. By Corollary 4.6 (b), there exists a predictable process $v = (v_t)_{t\geq 0}$ such that $1_{\{\sigma<\tau\}}1_{\{\sigma<\infty\}} = v_\sigma 1_{\{\sigma<\infty\}}$. Then $\mathrm{Gr}(\sigma_{\{\sigma<\tau\}}) = \{v = 1\} \cap \mathrm{Gr}(\sigma)$ is predictable, i.e., $\sigma_{\{\sigma<\tau\}}$ is predictable. □

A stopping time τ is *foretellable*, if there exists an increasing sequence of stopping times (τ_n) such that

$$\tau_n < \tau \text{ a.s. on } \{\tau > 0\} \text{ for all } n \text{ and } \tau_n \to \tau \text{ a.s. as } n \to \infty. \tag{14}$$

We abbreviate (14) by saying that (τ_n) *foretells* τ. We will denote the family of all foretellable stopping times by $\mathcal{T}_f$.

Proposition 4.9. *We have*

(a) $\tau \equiv s \in \overline{\mathbb{R}}_+ \implies \tau \in \mathcal{T}_f$,

(b) $(\tau_k) \subset \mathcal{T}_f, k \in \mathbb{N} \implies \bigwedge_{k=1}^m \tau_k \in \mathcal{T}_f,\ \bigvee_{k=1}^\infty \tau_k \in \mathcal{T}_f$,

(c) $(\tau_k) \subset \mathcal{T}_f, k \in \mathbb{N}$, *is stationary* $\implies \bigwedge_{k=1}^\infty \tau_k \in \mathcal{T}_f$,

(d) $\sigma \in \mathcal{T}_f$, *and* τ *is a stopping time such that* $\tau = \sigma$ *a.s.* $\implies \tau \in \mathcal{T}_f$,

(e) $\sigma, \tau \in \mathcal{T}_f \implies \sigma_{\{\sigma<\tau\}} \in \mathcal{T}_f$.

Proof. By Proposition 4.1 and Remark 4.2, all of the times in (a)-(e) are stopping times.

(a) is evident.

(b) Assume that τ_k is foretold by $(\tau_{k,n})_{n\geq 1}$. Then $\bigwedge_{k=1}^m \tau_k$ and $\bigvee_{k=1}^\infty \tau_k$ are foretold by $(\bigwedge_{k=1}^m \tau_{k,n})_{n\geq 1}$ and $(\bigwedge_{k=1}^\infty \tau_{k,n})_{n\geq 1}$, respectively.

(c) Denote $\tau = \bigwedge_{k=1}^\infty$ and $(\sigma_{k,m})$ a sequence of stopping times foretelling τ_k. For every k take a subsequence $(\sigma_{k,l}) \subset (\sigma_{k,m})$ such that

$$\mathbf{P}(e^{-\sigma_{k,l}} - e^{-\tau_k} > \frac{1}{2^k}) \leq \frac{1}{2^{k+l}}.$$

Put $\sigma_l = \inf_k \sigma_{k,l}$. Then (σ_l) is an increasing sequence of stopping times such that $\sigma_l \leq \tau$ for all l. On $\{\tau > 0\}$, we have $\tau_k > 0$ for all k, and therefore $\sigma_{k,l} < \tau_k$ a.s. for all k. So, by the assumption, we have, for all l, $\sigma_l < \tau$ a.s. on $\{\tau > 0\}$. Let $\sigma = \lim_{l\to\infty} \sigma_l$. We have, for all l,

$$\begin{aligned}\mathbf{P}(e^{-\sigma} - e^{-\tau} > \frac{1}{2^l}) &\leq \mathbf{P}(e^{-\sigma_l} - e^{-\tau} > \frac{1}{2^l}) \leq \mathbf{P}\big(\bigcup_{k=1}^\infty \{e^{-\sigma_{k,l}} - e^{-\tau} > \frac{1}{2^l}\}\big) \\ &\leq \mathbf{P}\big(\bigcup_{k=1}^\infty \{e^{-\sigma_{k,l}} - e^{-\tau_k} > \frac{1}{2^l}\}\big) \leq \sum_{k=1}^\infty \mathbf{P}(e^{-\sigma_{k,l}} - e^{-\tau_k} > \frac{1}{2^l}) \leq \frac{1}{2^l}.\end{aligned}$$

Hence, $\sigma = \tau$ a.s., and (σ_l) foretells τ.

(d) Suppose (σ_n) is a sequence of stopping times which foretells σ. Then $(\sigma_n \wedge \tau)$ foretells τ.

(e) Let (σ^n) and (τ^n) be two sequences of stopping times foretelling σ and τ, respectively. Put

$$\tilde{\sigma}_{m,n} = n \wedge \sigma^n_{\{\sigma^n<\tau^m\}}.$$

Then, by Proposition 4.1, $\tilde{\sigma}_{m,n}$ is a stopping time for every m, n. Moreover, for every fixed m, $(\tilde{\sigma}_{m,n})_{n\geq 1}$ foretells $\tilde{\sigma}_m = \sigma_{\{\sigma\leq\tau^m\}\cap\{\tau^m>0\}}$. Since, $\{\tau^m > 0\} \in \mathcal{F}_0$, $\{\sigma \leq \tau^m\} = \{\tau^m < \sigma\}^c$, $\{\tau^m < \sigma\} = \bigcup_{q\in\mathbb{Q}_+} \big(\big(\bigcup_{n\in\mathbb{N}}\{\tau^m \leq q - 1/n\}\big) \cap \{q < \sigma\}\big) \in \mathcal{F}_{\sigma-} \subset \mathcal{F}_\sigma$, we have

$$\{\sigma \leq \tau^m\} \cap \{\tau^m > 0\} \in \mathcal{F}_\sigma.$$

For each $t \geq 0$, we have $\{\sigma_{\{\sigma\leq\tau^m\}\cap\{\tau^m>0\}} \leq t\} \in \mathcal{F}_\sigma$, and $\sigma \leq \sigma_{\{\sigma\leq\tau^m\}\cap\{\tau^m>0\}}$. Thus,

$$\{\sigma_{\{\sigma\leq\tau^m\}\cap\{\tau^m>0\}} \leq t\} = \{\sigma_{\{\sigma\leq\tau^m\}\cap\{\tau^m>0\}} \leq t\} \cap \{\sigma \leq \tau^m\} \cap \{\tau^m > 0\} \in \mathcal{F}_t,$$

for each $t \geq 0$. This means that $\tilde{\sigma}_m = \sigma_{\{\sigma\leq\tau^m\}\cap\{\tau^m>0\}}$ is a stopping time, for each m. Put $\tilde{\sigma} = \bigwedge_{m=1}^\infty \tilde{\sigma}_m$. Then, by (c), $\tilde{\sigma}$ is foretellable, and since $\tilde{\sigma} = \sigma_{\{\sigma<\tau\}}$ a.s., by (d), $\sigma_{\{\sigma<\tau\}}$ is foretellable. □

Proposition 4.10. *Assume (τ_n) is a sequence of stopping times foretelling τ. Then there exists a negligible set $N \subset \Omega$ such that*

$$(\Omega \setminus N) \cap \mathcal{F}_{\tau-} = (\Omega \setminus N) \cap \bigvee_n \mathcal{F}_{\tau_n}.$$

Proof. There exists a negligible set N such that for each n, we have $\tau_n \leq \tau$ on $\Omega \setminus N$, and $\tau_n < \tau$ on $(\Omega \setminus N) \cap \{\tau > 0\}$. Denote $\tilde{\Omega} = \Omega \setminus N$. If $A \in \tilde{\Omega} \cap \mathcal{F}_{\tau_n}$, then $A \cap \{\tau_n = 0\} \in \tilde{\Omega} \cap \mathcal{F}_0 \subset \tilde{\Omega} \cap \mathcal{F}_{\tau-}$ and $A = (A \cap \{\tau_n < \tau\}) \cup (A \cap \{\tau = 0\}) \in \tilde{\Omega} \cap \mathcal{F}_{\tau-}$. Hence, $\tilde{\Omega} \cap \bigvee_n \mathcal{F}_{\tau_n} \subset \tilde{\Omega} \cap \mathcal{F}_{\tau-}$. On the other hand, if $A \in \tilde{\Omega} \cap \mathcal{F}_t$, $t \geq 0$, then $A \cap \{t < \tau_n\} \in \tilde{\Omega} \cap \mathcal{F}_{\tau_n-}$, and $A \cap \{t < \tau\} = \bigcup_n (A \cap \{t < \tau_n\}) \in \tilde{\Omega} \cap \bigvee_n \mathcal{F}_{\tau_n-}$. So, we have $\tilde{\Omega} \cap \mathcal{F}_{\tau-} \subset \tilde{\Omega} \cap \bigvee_n \mathcal{F}_{\tau_n-} \subset \tilde{\Omega} \cap \bigvee_n \mathcal{F}_{\tau_n}$. Thus, $\tilde{\Omega} \cap \mathcal{F}_{\tau-} = \tilde{\Omega} \cap \bigvee_n \mathcal{F}_{\tau_n}$, i.e.,

$$(\Omega \setminus N) \cap \mathcal{F}_{\tau-} = (\Omega \setminus N) \cap \bigvee_n \mathcal{F}_{\tau_n}.$$

□

Let $A \subset \Omega \times \mathbb{R}_+$. Put

$$D_A(\omega) = \inf\{t \in \mathbb{R}_+ : (\omega, t) \in A\},\ \omega \in \Omega.$$

We call D_A the *debut* of A.

Lemma 4.11. *Let $(\Omega, \mathcal{F})$ be a measurable space. For every $A \in \mathcal{A}(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+))$, the debut D_A is $\widehat{\mathcal{F}}$-measurable.*

Proof. Let $r > 0$. Then $\{D_A < r\}$ is the projection of $A \cap (\Omega \times [0, r[)$ onto Ω. Hence, by Theorem 3.3 and Remark 3.4, $\{D_A < r\} \in \mathcal{A}(\mathcal{F})$. By Theorem 3.10, D_A is $\widehat{\mathcal{F}}$-measurable. □

Lemma 4.12. *Put*

$$\mathcal{C}^o = \{\bigcup_{k=1}^n [\sigma_k, \tau_k[: \sigma_k \leq \tau_k,\ \sigma_k, \tau_k \in \mathcal{T}\},$$
$$\mathcal{C}^p = \{\bigcup_{k=1}^n [\sigma_k, \tau_k[: \sigma_k \leq \tau_k,\ \sigma_k, \tau_k \in \mathcal{T}_f \cap \mathcal{T}_p\}.$$

Then $\mathcal{C}^o$ is an algebra on $\Omega \times \mathbb{R}_+$ generating $\mathcal{O}$, and $\mathcal{C}^p$ is an algebra on $\Omega \times \mathbb{R}_+$ generating $\mathcal{P}$.

Proof. The fact that $\mathcal{C}^o$ is an algebra on $\Omega\times\mathbb{R}_+$ follows from the properties (a) and (b) of Proposition 4.1, and $\mathcal{C}^o$ is an algebra on $\Omega\times\mathbb{R}_+$ by the properties (a) and (b) of Properties 4.8 and Proposition 4.9.

By Proposition 4.3, $\mathcal{O}=\sigma(\mathcal{C})$, where $\mathcal{C}=\{[\tau,\infty[\colon \tau\in\mathcal{T}\}$. We have $\mathcal{C}\subset\mathcal{C}^o\subset\sigma(\mathcal{C})$. Hence, $\sigma(\mathcal{C}^o)=\mathcal{O}$.

By Proposition 4.4, $\mathcal{P}=\sigma(\mathcal{C})$, where $\mathcal{C}=\{A\times\{0\}:A\in\mathcal{F}_0\}\cup\{A\times[s,t[\colon 0<s<t,\ s,t,\in\mathbb{Q}_+,\ A\in\bigcup_{r<s}\mathcal{F}_r\}$. If $A\in\mathcal{F}_0$, then $A\times\{0\}=\bigcap_{n=1}^{\infty}[0_A,(1/n)_A[\in\sigma(\mathcal{C}^p)$. The stopping times 0_A and $(1/n)_A$ areforetellable by sequences $(k\wedge 0_A)_{k\geq1}$ and $(k\wedge(1/n-1/(n+k)))_{k\geq1}$, respectively. If $A\in\mathcal{F}_r$, $r<s$, then $A\times[s,t[=[s_A,t_A[\in\sigma(\mathcal{C}^p)$. Choose n such that $A\in\mathcal{F}_{s-1/n}$. The stopping times s_A and t_A areforetellable by sequences $((n+k)\wedge(s-1/(n+k))_A)_{k\geq1}$ and $((n+k)\wedge(t-1/(n+k))_A)_{k\geq1}$, respectively. We have $\mathcal{C}\subset\mathcal{C}^p\subset\sigma(\mathcal{C})$. Hence, $\sigma(\mathcal{C}^p)=\mathcal{P}$. □

Remark 4.13. One can replace $\mathcal{T}_f\cap\mathcal{T}_p$ with $\mathcal{T}_p$ in Lemma 4.12.

Remark 4.14. We have $\mathcal{T}_p\subset\mathcal{T}$. By Lemma 4.12, $\mathcal{P}\subset\mathcal{O}$.

Theorem 4.15. *Put*

$$\mathcal{C}^o=\{\bigcup_{k=1}^{n}[\sigma_k,\tau_k[\colon \sigma_k\leq\tau_k,\ \sigma_k,\tau_k\in\mathcal{T}\}.$$

Then for any $A\in\mathcal{C}^o_\delta$ *we have* $\mathrm{Gr}(D_A)\in A$*, and there exists* $\tau\in\mathcal{T}$ *such that* $\tau=D_A$ *a.s..*

Proof. Let $A\in\mathcal{C}^o_\delta$. For each $\omega\in\Omega$, the set $\{t\geq0:(\omega,t)\in A\}$ is closed under taking limit from the right in $\mathbb{R}_+$. Hence, $\mathrm{Gr}(D_A)\subset A$. Put

$$\mathcal{H}=\{\sigma\in\mathcal{T}:\sigma\leq D_A\}. \tag{15}$$

By Proposition 2.5, there exists a sequence $(\sigma_n)\subset\mathcal{H}$ such that $\bigvee_n\sigma_n=\operatorname{ess\,sup}\mathcal{H}$. Put

$$\tau=\operatorname{ess\,sup}\mathcal{H}.$$

By Proposition 4.1 (b), we have $\tau\in\mathcal{T}$. We will show that $\tau=D_A$ *a.s.*. Let $(A_n)\subset\mathcal{C}^o$ be a decreasing sequence such that $A=\bigcap_{n=1}^{\infty}A_n$. Put

$$B_n=A_n\cap[\tau,\infty[.$$

Then $(B_n)\subset\mathcal{C}^o$, and $B_n\downarrow A\cap[\tau,\infty[=A$. For arbitrary $\sigma,\tau\in\mathcal{T}$, we have $D_{[\sigma,\tau[}=\sigma_{\{\sigma<\tau\}}$, so by Proposition 4.1 (c), $D_{[\sigma,\tau[}\in\mathcal{T}$. Suppose $B_n=\bigcup_{k=1}^{m}B_{n,k}=\bigcup_{k=1}^{m}[\sigma_{n,k},\tau_{n,k}[$. Then, by Proposition 4.1 (b), $D_{B_n}=\bigwedge_{k=1}^{m}D_{B_{n,k}}\in\mathcal{T}$, and $D_{B_n}\geq\tau$. Since $B_n\supset A$, we have $D_{B_n}\leq D_A$. Therefore, $D_{B_n}\in\mathcal{H}$. By Proposition 2.5, we must have $D_{B_n}=\tau$ a.s. for each n. Since $\mathrm{Gr}(D_{B_n})\subset B_n$, we have $\mathrm{Gr}(\tau)\subset\bigcap_{n=1}^{\infty}B_n=A$ for almost all $\omega\in\Omega$. Thus, $\tau\geq D_A$ a.s.. Since we have already shown that $\tau\leq D_A$, we have $\tau=D_A$ a.s.. □

Theorem 4.16. *Put*

$$\mathcal{C}^p = \{\bigcup_{k=1}^{n}[\sigma_k, \tau_k[: \sigma_k \leq \tau_k,\ \sigma_k, \tau_k \in \mathcal{T}_p\}.$$

Then for any $A \in \mathcal{C}^p_\delta$ we have $\mathrm{Gr}(D_A) \in A$, and there exists $\tau \in \mathcal{T}_p$ such that $\tau = D_A$ a.s..

Proof. Copy the proof of Theorem 4.15. Then replace $\mathcal{T}$ by $\mathcal{T}_p$, $\mathcal{C}^o$ by $\mathcal{C}^p$ and Proposition 4.1 by Properties 4.8. □

Theorem 4.17. *Put*

$$\mathcal{C}^f = \{\bigcup_{k=1}^{n}[\sigma_k, \tau_k[: \sigma_k \leq \tau_k,\ \sigma_k, \tau_k \in \mathcal{T}_f\}.$$

Then for any $A \in \mathcal{C}^f_\delta$ we have $\mathrm{Gr}(D_A) \in A$, and there exists $\tau \in \mathcal{T}_f$ such that $\tau = D_A$ a.s..

Proof. Copy the proof of Theorem 4.15. Then replace $\mathcal{T}$ by $\mathcal{T}_f$, $\mathcal{C}^o$ by $\mathcal{C}^f$ and Proposition 4.1 (a), (b) and (c) by Proposition 4.9 (a), (b) and (e), respectively. □

4.2 Section Theorems

Let $A \subset \Omega \times \mathbb{R}_+$ and τ be an $\mathcal{F}$-measurable non-negative random variable. If

$$\tau(\omega) < \infty \implies (\omega, \tau(\omega)) \in A, \tag{16}$$

we say that τ is a *section* of A. If, in addition,

$$\mathbf{P}(\tau < \infty) = \mathbf{P}(D_A < \infty), \tag{17}$$

we say that τ is a *full section* of A.

Lemma 4.18. (Section lemma). *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $A \in \mathcal{A}(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+))$. Then there exists a full section of A.*

Proof. Put

$$\overline{\mathbf{P}}(D) = \inf\{\mathbf{P}(E) : E \in \mathcal{F}, D \subset E\},\ D \subset \Omega.$$

Then $\overline{\mathbf{P}}$ is an $\mathcal{F}$-capacity on Ω. Let π denote the projection from $\Omega \times \mathbb{R}_+$ onto Ω. Put

$$I(C) = \overline{\mathbf{P}}(\pi_\Omega(C)),\ C \subset \Omega \times \mathbb{R}_+.$$

Then, by Lemma 3.7, I is a $\mathcal{G}$-capacity on $\Omega \times \mathbb{R}_+$, where $\mathcal{G} = \{\bigcup_{k=1}^n G_k,\ G_k \in \mathcal{F} \times \mathcal{K}(\mathbb{R}_+)\}$.

Since $A \in \mathcal{A}(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+))$, by Theorem 3.8, for any given $\epsilon > 0$ there exists $B \in \mathcal{G}_\delta$, $B \subset A$ such that $I(B) > I(A) - \epsilon$, i.e.,

$$\mathbf{P}(D_B < \infty) > \mathbf{P}(D_A < \infty) - \epsilon.$$

By Lemma 4.11, the debut D_B is $\widehat{\mathcal{F}}$-measurable, so there exists an $\mathcal{F}$-measurable random variable σ^ϵ such that $\sigma^\epsilon = D_B$ a.s.. Let $F \in \mathcal{F}$, $F \subset \{\sigma^\epsilon = D_B\}$ such that $\mathbf{P}(F) = 1$. Let $\tau^\epsilon = \sigma^\epsilon_F$. Then τ^ϵ is $\mathcal{F}$-measurable, $\tau^\epsilon = \sigma^\epsilon = D_B$ a.s., and $\tau^\epsilon = D_B$ on $\{\tau < \infty\}$. Since $B \in \mathcal{G}_\delta$, we have that $\{t \geq 0 : (\omega, t) \in B\}$ is a compact set. Hence, $D_B(\omega) < \infty \implies (\omega, D_B(\omega)) \in B \subset A$. We have that τ^ϵ is a section of A, and

$$\mathbf{P}(\tau^\epsilon < \infty) = \mathbf{P}(D_B < \infty) > \mathbf{P}(D_A < \infty) - \epsilon.$$

In particular, we may choose $\epsilon > 0$ such that

$$\mathbf{P}(\tau^\epsilon < \infty) \geq \frac{1}{2}\mathbf{P}(D_A < \infty).$$

We will construct recursively a sequence of sections of A converging to a full section of A. Put $\tau_0 = \infty$. Assume τ_n is defined. Put

$$A_n = A \cap (\{\tau_n = \infty\} \times \mathbb{R}_+).$$

Clearly $A_n \in \mathcal{A}(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+))$, and above we showed that then there exists a section σ_n of A_n such that

$$\mathbf{P}(\sigma_n < \infty) \geq \frac{1}{2}\mathbf{P}(D_{A_n} < \infty) = \frac{1}{2}\mathbf{P}(\{\tau_n = \infty\} \cap \{D_A < \infty\}).$$

Let $\tau_{n+1} = \tau_n \wedge \sigma_n$, then τ_{n+1} is a section of A and

$$\begin{aligned} \mathbf{P}(\tau_{n+1} < \infty) &= \mathbf{P}(\tau_n < \infty) + \mathbf{P}(\sigma_n < \infty) \\ &\geq \mathbf{P}(\tau_n < \infty) + \frac{1}{2}\mathbf{P}(\{\tau_n = \infty\} \cap \{D_A < \infty\}). \end{aligned} \tag{18}$$

Set $\tau = \lim_{n\to\infty} \tau_n$. Since $\tau_{n+1}1_{\{\tau_n<\infty\}} = \tau_n 1_{\{\tau_n<\infty\}}$, we have $\tau 1_{\{\tau_n<\infty\}} = \tau_n 1_{\{\tau_n<\infty\}}$, and therefore $\{\tau < \infty\} = \bigcup_{n=1}^\infty \{\tau_n < \infty\}$ and $\{\tau = \infty\} = \bigcap_{n=1}^\infty \{\tau_n = \infty\}$. By letting $n \to \infty$ in (18) we get

$$\mathbf{P}(\tau < \infty) \geq \mathbf{P}(\tau < \infty) + \frac{1}{2}\mathbf{P}(\{\tau = \infty\} \cap \{D_A < \infty\}).$$

Hence, $\mathbf{P}(\{\tau = \infty\} \cap \{D_A < \infty\}) = 0$, i.e., $\{D_A < \infty\} \subset \{\tau < \infty\}$ a.s.. But from $\{\tau < \infty\} = \bigcup_{n=1}^\infty \{\tau_n < \infty\}$, we have that τ is a section of A, and $\{\tau < \infty\} \subset \{D_A < \infty\}$. Therefore, $\{\tau < \infty\} = \{D_A < \infty\}$ a.s., i.e., τ is a full section of A. □

Lemma 4.19. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $\mathcal{S}$ be a sub-σ-algebra of $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$, and $\mathcal{C}$ an algebra generating $\mathcal{S}$. Let $A \in \mathcal{A}(\mathcal{S})$. Then for every given $\epsilon > 0$ there exists $B \in \mathcal{C}_\delta$ such that*

$$B \subset A,$$

$$\mathbf{P}(\pi(B)) > \mathbf{P}(\pi(A)) - \epsilon.$$

Proof. By Lemma 4.18, there exists a full section τ of A. Let us define a measure μ on $\mathcal{S}$ as follows:

$$\mu(S) = \mathbf{P}(\{\omega : (\omega, \tau(\omega)) \in S\}), \ S \in \mathcal{S}.$$

Put

$$I(C) = \inf\{\mu(D) : D \in \mathcal{S}, C \subset D\}, \ C \subset \Omega \times \mathbb{R}_+.$$

Then I is a $\mathcal{S}$-capacity on $\Omega \times \mathbb{R}_+$. Since $A \in \mathcal{A}(\mathcal{S})$, by Theorem 3.8, for any given $\epsilon > 0$, there exists $B' \in \mathcal{S}$, $B' \subset A$ such that $\mu(B') > I(A) - \epsilon/2$, and by Proposition 2.4, there exists $B \in \mathcal{C}_\delta$, $B \subset B'$ such that $\mu(B) \geq \mu(B') - \epsilon/2$. On the other hand, for any $B \in \mathcal{S}$ we have $\pi(B) \supset \{\omega : (\omega, \tau(\omega)) \in B\}$. Hence, $\mathbf{P}(\pi(B)) \geq \mu(B)$. We obtain

$$\mathbf{P}(\pi(B)) \geq \mu(B) \geq \mu(B') - \epsilon/2 > I(A) - \epsilon \geq \mathbf{P}(\pi(A)) - \epsilon.$$

□

Theorem 4.20. (Optional section theorem). *Let A be an $\mathcal{O}$-analytic set. For any given $\epsilon > 0$ there exists a stopping time τ such that*

(a) $\mathrm{Gr}(\tau) \subset A$,

(b) $\mathbf{P}(\{\tau < \infty\}) > \mathbf{P}(\pi(A)) - \epsilon$.

Proof. Put $\mathcal{C}^o = \{\bigcup_{k=1}^n [\sigma_k, \tau_k[: \sigma_k \leq \tau_k, \ \sigma_k, \tau_k \in \mathcal{T}\}$. By Lemma 4.12, $\mathcal{C}^o$ is an algebra and $\sigma(\mathcal{C}^o) = \mathcal{O}$, by the assumption $A \in \mathcal{A}(\mathcal{O})$. Then, by Lemma 4.19, for any given $\epsilon > 0$ there exists $B \in \mathcal{C}^o_\delta$ such that $B \subset A$ and $\mathbf{P}(\pi(B)) > \mathbf{P}(\pi(A)) - \epsilon$. By Theorem 4.15, there exists a stopping time σ such that $\sigma = D_B$ a.s.. Put

$$C = \{\omega : (\omega, \sigma(\omega))\}.$$

Then $1_C = 1_B(\sigma)1_{\{\tau<\infty\}}$, and, by Theorem 4.6 (a), $C \in \mathcal{F}_\sigma$. By Theorem 4.15, $\mathrm{Gr}(D_B) \subset B$, and we have $\mathbf{P}(C \cup \{\sigma = \infty\}) = 1$. Set $\tau = \sigma_C$. Since $\{\tau \leq t\} = C \cap \{\sigma \leq t\}$, τ is a stopping time. Moreover, we have $\mathrm{Gr}(\tau) \subset B \subset A$, and $\tau = \sigma = D_B$ a.s.. Hence,

$$\mathbf{P}(\{\tau < \infty\}) = \mathbf{P}(\{D_B < \infty\}) = \mathbf{P}(\pi(B)) > \mathbf{P}(\pi(A)) - \epsilon.$$

□

Theorem 4.21. (Predictable section theorem). *Let A be a $\mathcal{P}$-analytic set. For any given $\epsilon > 0$ there exists a predictable stopping time τ such that*

(a) $\mathrm{Gr}(\tau) \subset A$,

(b) $\mathbf{P}(\{\tau < \infty\}) > \mathbf{P}(\pi(A)) - \epsilon$.

Proof. The proof is essentially the same as the one given for Theorem 4.20; the only difference being that here $C \in \mathcal{F}_{\sigma-}$ instead of $\mathcal{F}_\sigma$, and to see that the restriction of σ on C is predictable we use Theorem 4.6 (a) to conclude that there exists a predictable process v such that $1_C 1_{\{\sigma<\infty\}} = v_\sigma 1_{\{\sigma<\infty\}}$, and since $\mathrm{Gr}(\sigma_C) = \{v = 1\} \cap \mathrm{Gr}(\sigma)$ is predictable, σ_C is predictable. □

Theorem 4.22. *Put*

$$\mathcal{C}^f = \{\bigcup_{k=1}^{n} [\sigma_k, \tau_k[: \sigma_k \leq \tau_k,\ \sigma_k, \tau_k \in \mathcal{T}_f\}.$$

Let $A \in \mathcal{A}(\sigma(\mathcal{C}^f))$, then for any given $\epsilon > 0$ there exists an foretellable stopping time τ such that

(a) $\mathrm{Gr}(\tau) \subset A$,

(b) $\mathbf{P}(\{\tau < \infty\}) > \mathbf{P}(\pi(A)) - \epsilon$.

Proof. One can use the same procedure as in the proof of Theorem 4.20 to construct the section. See the construction of stopping times σ and τ in the proof of Theorem 4.20. If Theorem 4.17 is used instead of Theorem 4.15, then $\tau = \sigma$ a.s., where τ a stopping time satisfying (a) and (b), and σ is foretellable. By Proposition 4.9 (d), τ is foretellable. □

Remark 4.23. In the literature, Lemma 4.19 and therefore also Theorem 4.15, Theorem 4.16 and Theorem 4.17 are given in the weaker form, where the set A is allowed to be only measurable, not analytic.

Corollary 4.24. *Let τ be a stopping time. If τ is predictable, then τ is foretellable.*

Proof. Let $\mathcal{C}^f$ be as in Theorem 4.22. By Lemma 4.12, we have $\mathcal{P} \subset \sigma(\mathcal{C}^f)$. Let τ be a predictable stopping time. Then $\mathrm{Gr}(\tau) \in \mathcal{P}$, and therefore $\mathrm{Gr}(\tau) \in \sigma(\mathcal{C}^f)$. By Theorem 4.22, for any given $\epsilon > 0$ there exists an foretellable stopping time σ^ϵ such that $\mathrm{Gr}(\sigma^\epsilon) \subset \mathrm{Gr}(\tau)$ and $\mathbf{P}(\{\sigma^\epsilon < \infty\}) \geq \mathbf{P}(\tau < \infty) - \epsilon$. For $n \geq 2$, define

$$\tau_n = \bigwedge_{k=2}^{n} \sigma^{1/k}.$$

We have $\tau = \lim_{\to\infty} \tau_n$ a.s., and, by Proposition 4.9 (c) and (d), τ is foretellable. □

4.3 Martingales

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration of $\mathcal{F}$ satisfying the usual conditions. A subset E of $\Omega \times \mathbb{R}_+$ is said to be *evanescent* (w.r.t. $\mathbf{P}$), provided that the projection $\pi(E)$ onto Ω is a $\mathbf{P}$-null set. Two processes $u = (u_t)$ and $v = (v_t)$ are said to be *modifications* of each other, if, for each $t \geq 0$, $u_t = v_t$ a.s., and $\mathbf{P}$-*indistinguishable*, if $\{(\omega, t) : u_t(\omega) \neq v_t(\omega)\}$ is an $\mathbf{P}$-evanescent set. We abbreviate u and v being indistinguishable by writing $u = v$.

For a fixed $\omega \in \Omega$, a mapping $v(\omega) : t \mapsto v_t(\omega)$ is called a trajectory of v. If u and v are indistinguishable, then for almost all ω they have identical trajectories. It is clear that, if u and v are indistinguishable, then they are modifications of each other, but, without an additional continuity assumption, the two notions are not equal.

Let I be a well ordered subset of $\overline{\mathbb{R}}$, and $(\mathcal{F}_i)_{i\in I}$ a filtration of $\mathcal{F}$. An $(\mathcal{F}_i)_{i\in I}$-adapted process $v = (v_i)_{i\in I}$ is called a $(\mathcal{F}_i)_{i\in I}$-*martingale*, if for each $i \in I$, v_i is integrable, and

$$\mathbf{E}[v_t|\mathcal{F}_s] = v_s \text{ a.s. for all } s \le t,\ s,t \in I. \tag{19}$$

The suffix "$(\mathcal{F}_i)_{i\in I}$-" is often omitted and we simply say that v is a martingale.

Recall the notation $\mathbb{N} = \{0, 1, 2, ...\}$, $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, $\mathbb{N}_- = \{\cdots, -2, -1, 0\}$. If I is denumerable, say $I = \mathbb{N}$, then (19) is equivalent with $\mathbf{E}[v_{n+1}|\mathcal{F}_n] = v_n$ a.s. for every $n \in \mathbb{N}$. We begin by the martingales indexed over denumerable I, and work towards the martingales indexed over $\overline{\mathbb{R}}_+ = [0, +\infty]$.

Lemma 4.25. *Let $v = (v_n)_{n\in\mathbb{N}_-}$ be a martingale. Then (v_n) is uniformly integrable.*

Proof. By the tower property of conditional expectation, we have that $\mathbf{E}[v_0|\mathcal{F}_0] = v_n$ a.s. for all $n \in \mathbb{N}_-$. Since v_0 is integrable, by Theorem 2.1, v is uniformly integrable. □

Theorem 4.26. *Let $v = (v_n)$, $n \in \mathbb{N}$, be a martingale, σ, τ be two bounded stopping times, and $\sigma \le \tau$. Then v_σ and v_τ are integrable, and*

$$\mathbf{E}[v_\tau|\mathcal{F}_\sigma] = v_\sigma \textit{ a.s.}.$$

Proof. Suppose $\tau \le N$. Then $|v_\tau| \le \sum_{k=0}^N |v_k|$, $|v_\sigma| \le \sum_{k=0}^N |v_k|$, and therefore v_τ, v_σ are integrable. Assume $A \in \mathcal{F}_\sigma$. For every $n \in \mathbb{N}$, we have

$$A \cap \{\sigma = n\} \cap \{\tau > n\} \in \mathcal{F}_n.$$

Suppose $\tau - \sigma \le 1$. We have

$$\int_A (v_\tau - v_\sigma)\mathbf{P}(d\omega) = \sum_{n=0}^N \int_{A\cap\{\sigma=n\}\cap\{\tau>n\}} (v_{n+1} - v_n)\mathbf{P}(d\omega) = 0. \tag{20}$$

Put $\tau_n = \tau \wedge (\sigma + n)$, $n = 1, \cdots, N$. Then each τ_n is a stopping time, and $\sigma \le \tau_1 \le \cdots \le \tau_N = \tau$, where $\tau_1 - \sigma \le 1$, and $\tau_{n+1} - \tau_n \le 1$ for $n = 1, \ldots, N-1$. Let $A \in \mathcal{F}_\sigma$. From (20), we have

$$\int_A (v_{\tau_1} - v_\sigma)\mathbf{P}(d\omega) = 0 \text{ and } \int_A (\tau_{n+1} - \tau_n)\mathbf{P}(d\omega) = 0 \text{ for } n = 1, \cdots, N.$$

We obtain $\int_A (v_\tau - v_\sigma)\mathbf{P}(d\omega)$ for every $A \in \mathcal{F}_\sigma$, i.e., $\mathbf{E}[v_\tau|\mathcal{F}_\sigma] = v_\sigma$ a.s... □

Theorem 4.27. *Let $v = (v_n)$, $n \in \overline{\mathbb{N}}$, be a martingale, σ, τ be two stopping times, and $\sigma \le \tau$. Then v_σ and v_τ are integrable, and*

$$\mathbf{E}[v_\tau|\mathcal{F}_\sigma] = v_\sigma \textit{ a.s.}.$$

Proof. The set $\{0, 1, \cdots, n, \infty\}$ is an order preserving isomorphism of $\{0, 1, \cdots, n, n+1\}$. Put $\sigma_n = \sigma_{\{\sigma \le n\}}$. By Theorem 4.26, we have $\mathbf{E}[v_\infty|\mathcal{F}_{\sigma_n}] = v_{\sigma_n}$ a.s.. Since $\mathcal{F}_\sigma \cap \{\sigma = \sigma_n\} = \mathcal{F}_{\sigma_n} \cap \{\sigma = \sigma_n\}$, by Proposition 2.3 (c)

$$\mathbf{E}[v_\infty|\mathcal{F}_\sigma]1_{\{\sigma=\sigma_n\}} = \mathbf{E}[v_\infty|\mathcal{F}_{\sigma_n}]1_{\{\sigma=\sigma_n\}} = v_{\sigma_n}1_{\{\sigma=\sigma_n\}} = v_\sigma 1_{\{\sigma=\sigma_n\}} \text{ a.s..}$$

As $\{\sigma = \sigma_n\} \uparrow \Omega$, we get $\mathbf{E}[v_\infty|\mathcal{F}_\sigma] = v_\sigma$ a.s.. Equally, we have for τ, that $\mathbf{E}[v_\infty|\mathcal{F}_\tau] = v_\tau$ a.s.. This means that v_σ and v_τ are integrable, and

$$\mathbf{E}[v_\tau|\mathcal{F}_\sigma] = \mathbf{E}[\mathbf{E}[v_\infty|\mathcal{F}_\tau]|\mathcal{F}_\sigma] = \mathbf{E}[v_\infty|\mathcal{F}_\sigma] = v_\sigma \text{ a.s..}$$

□

Let $v = (v_n)$, $n \in \mathbb{N}$, be an adapted process. We say that v *upcrosses* an interval $[a, b]$, if $v_s < a < b < v_t$ or $v_s < a < b < v_t$ for $s < t$. We denote by $U_a^b[v, N]$ the number of consecutive upcrossings of $[a, b]$ by $\{v_0, v_1, \cdots, v_N\}$.

Lemma 4.28. *Let $v = (v_n)$, $n \in \mathbb{N}$, be a martingale. Then for any $\lambda > 0$, $N \ge 1$, $a, b \in \mathbb{R}$, $a < b$, we have*

$$\lambda \mathbf{P}(\sup_{n \le N} |v_n| \ge \lambda) \le \mathbf{E}[v_0] + 2\mathbf{E}[v_N^-]; \tag{21}$$

$$\mathbf{E}U_a^b[v, N] \le \frac{1}{b-a}\mathbf{E}[(v_N - a)^-]. \tag{22}$$

Proof. Put $\tau = \inf\{n : v_n \ge \lambda\} \wedge N$. Then τ is a bounded stopping time, and we have $v_\tau \ge \lambda$ on $\{\sup_{n \le N} v_n \ge \lambda\}$ and $\tau = N$ on $\{\sup_{n \le N} v_n < \lambda\}$. By Theorem 4.26,

$$\begin{aligned}\mathbf{E}[v_0] = \mathbf{E}[v_\tau] &= \int_{\{\sup_{n\le N} v_n \ge \lambda\}} v_\tau \mathbf{P}(d\omega) + \int_{\{\sup_{n\le N} v_n < \lambda\}} v_\tau \mathbf{P}(d\omega) \\ &\ge \lambda \mathbf{P}(\sup_{n \le N} v_n \ge \lambda) + \int_{\{\sup_{n\le N} v_n < \lambda\}} v_N \mathbf{P}(d\omega).\end{aligned} \tag{23}$$

Similarly, put $\tau = \inf\{n : v_n \le -\lambda\} \wedge N$. Then τ is a bounded stopping time, and we have $v_\tau \le -\lambda$ on $\{\inf_{n \le N} v_n \le -\lambda\}$ and $\tau = N$ on $\{\inf_{n \le N} v_n > -\lambda\}$. By Theorem 4.26, $\mathbf{E}[v_\tau] = \mathbf{E}[v_N]$, and we have

$$\int_{\{\inf_{n\le N} v_n \le -\lambda\}} v_N \mathbf{P}(d\omega) = \int_{\{\inf_{n\le N} v_n \le -\lambda\}} v_\tau \mathbf{P}(d\omega) \le -\lambda \mathbf{P}(\inf_{n \le N} v_n \le -\lambda). \tag{24}$$

Combining (23) and (24), we get

$$\begin{aligned}\lambda \mathbf{P}(\sup_{n \le N} |v_n| \ge \lambda) &= \lambda \mathbf{P}(\sup_{n \le N} v_n \ge \lambda) + \lambda \mathbf{P}(\inf_{n \le N} v_n \le \lambda) \\ &\le \mathbf{E}[v_0] - \int_{\{\sup_{n\le N} v_n < \lambda\}} v_N \mathbf{P}(d\omega) - \int_{\{\inf_{n\le N} v_n \le -\lambda\}} v_N \mathbf{P}(d\omega) \\ &\le \mathbf{E}[v_0] + 2\mathbf{E}[v_N^-].\end{aligned}$$

So, (21) holds. Let us next show (22).

Put $\tau_0 = \inf\{n : v_n \le a\}$, and, for $k \ge 1$, define

$$\tau_k = \begin{cases} \inf\{n : n > \tau_{k-1}, v_n \le a\}, & \text{if } n \text{ is even,} \\ \inf\{n : n > \tau_{k-1}, v_n \ge b\}, & \text{if } n \text{ is odd.} \end{cases}$$

Then (τ_k) is a sequence of stopping times, and by Theorem 4.26, we have for every $k \ge 0$

$$\begin{aligned} 0 = \mathbf{E}[v_{\tau_{2k+1}\wedge N} - v_{\tau_{2k}\wedge N}] &= \mathbf{E}[v_{\tau_{2k+1}\wedge N} - v_{\tau_{2k}\wedge N}(1_{\{\tau_{2k}\le N<\tau_{2k+1}\}} + 1_{\{N\ge\tau_{2k+1}\}})] \\ &\ge \mathbf{E}[(v_N - a)1_{\{\tau_{2k}\le N<\tau_{2k+1}\}} + (b-a)1_{\{N\ge\tau_{2k+1}\}}]. \end{aligned} \tag{25}$$

Since $\{U_a^b[v,N] = k\} = \{\tau_{2k} \le N < \tau_{2k+1}\}$, we have $\{U_a^b[v,N] \ge k+1\} \subset \{N \ge \tau_{2k+1}\}$, and $\{\tau_{2k} \le N < \tau_{2k+1}\} \subset \{U_a^b[v,N] = k\}$. So, from (25) we get

$$\mathbf{P}(U_a^b[v,N] \ge k+1) \le \frac{1}{b-a}\mathbf{E}[(v_N - a)^- 1_{\{U_a^b[v,N]=k\}}].$$

By $\mathbf{E}U_a^b[v,N] = \sum_{k=0}^\infty \mathbf{P}(U_a^b[v,N] \ge k+1)$, we get (22), which completes the proof. $\square$

Let $v = (v_t)$, $t \in \mathbb{R}_+$, be an adapted process, and C be a countable subset of $\mathbb{R}_+$. Suppose $C = \{t_1, t_2, \cdots\}$ and $C_n = \{t_1, t_2, \cdots, t_n\}$. We denote by $U_a^b[v, C_n]$ the number of consecutive upcrossings of $[a,b]$ by $\{v_{t_1}, v_{t_2}, \cdots, v_{t_n}\}$, and define

$$U_a^b[v,C] = \lim_{n\to\infty} U_a^b[v, C_n].$$

Lemma 4.29. *Let $v = (v_t)$, $t \in \mathbb{R}_+$, be a martingale. Then for any $r, s \in \mathbb{R}_+$, $r < s$, $a, b \in \mathbb{R}$, $a < b$, and $\lambda > 0$ we have*

$$\lambda \mathbf{P}(\sup_{t\in\mathbb{Q}\cap[r,s]} |v_t| \ge \lambda) \le \mathbf{E}[v_r] + 2\mathbf{E}[v_s^-]; \tag{26}$$

$$\mathbf{E}U_a^b[v, \mathbb{Q}\cap[r,s]] \le \frac{1}{b-a}\mathbf{E}[(v_s - a)^-]. \tag{27}$$

Proof. The set $\mathbb{Q}\cap[r,s]$ is an order preserving isomorphism of $\mathbb{N}$. In Lemma 4.28, let $N \to \infty$. Then (21) yields (26), and respectively, (22) yields (27). $\square$

Proposition 4.30. *Let x be an integrable random variable, and $(\mathcal{F}_n)$ a filtration. Then*

$$\mathbf{E}[x|\mathcal{F}_n] \to \mathbf{E}[x|\mathcal{F}_\infty] \ \textit{a.s.}.$$

Proof. Denote $v_n = \mathbf{E}[x|\mathcal{F}_n]$, and $v = (v_n)$. By Proposition 2.1, v is a uniformly integrable martingale. Let $a, b \in \mathbb{Q}$, $a < b$. By Lemma 4.28, we have

$$\lim_{N\to\infty} \mathbf{E}U_a^b[v,N] \le \frac{1}{b-a}\sup_N \mathbf{E}[(v_N - a)^-] < \infty.$$

Hence, $\lim_{N\to\infty} \mathbf{E}U_a^b[v,N] < \infty$ a.s.. Set

$$A_{a,b} = \{\liminf_{n\to\infty} v_n < a,\ \limsup_{n\to\infty} v_n > b\},\ \text{and}\ A = \bigcup_{a,b\in\mathbb{Q},a<b} A_{a,b}.$$

Since $A_{a,b} \subset \{\lim_{N\to\infty} \mathbf{E}U_a^b[v,N] = \infty\}$, we have $\mathbf{P}(A_{a,b}) = 0$, and hence $\mathbf{P}(A) = 0$. Put

$$v_\infty(\omega) = \begin{cases} \lim_{n\to\infty} v_n(\omega), & \text{if } \omega \notin A, \\ 0, & \text{otherwise.} \end{cases}$$

We have $v_n \to v_\infty$ a.s., and by Fatou's lemma, $\mathbf{E}[|v_\infty|] \le \sup_n \mathbf{E}[|v_n|] < \infty$, i.e., v_∞ is integrable. Let $A \in \bigcup_{n=1}^\infty \mathcal{F}_n$. Then $A \in \mathcal{F}_n$ for some n, and by Theorem 4.27, we have

$$\mathbf{E}[v_\infty 1_A] = \mathbf{E}[v_n 1_A] = \mathbf{E}[x 1_A] = \mathbf{E}[\mathbf{E}[x|\mathcal{F}_\infty] 1_A].$$

Since v_∞ and $\mathbf{E}[x|\mathcal{F}_\infty]$ are $\mathcal{F}_\infty$-measurable, and $\mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n)$, we have, by Corollary 10, $v_\infty = \mathbf{E}[x|\mathcal{F}_\infty]$ a.s.. $\square$

Lemma 4.31. (Föllmer's lemma). *Let $v = (v_t)$, $t \in \mathbb{R}_+$, be a martingale. Then v has an adapted right-continuous modification $\tilde{v}$. Moreover, $\tilde{v}$ is a martingale.*

Proof. Let $t \in \mathbb{R}_+$, $\mathbb{Q}_t = (\mathbb{Q} \cap [0,t]) \cup \{t\}$, $a, b \in \mathbb{Q}$, and $a < b$. Put

$$H_{t,a,b} = \{\omega : \sup_{q\in\mathbb{Q}_t} |v_q(\omega)| \vee U_a^b[v(\omega), \mathbb{Q}_t] = \infty\},$$

and

$$H_t = \bigcap_{s>t} \Big(\bigcup_{\substack{a,b\in\mathbb{Q} \\ a<b}} H_{s,a,b} \Big).$$

We have $H_t \in \mathcal{F}_{t+}$, but remember that we are assuming a right-continuous filtration, and therefore $H_t \in \mathcal{F}_t$. Furthermore, $H_t \uparrow H$, where $H = \bigcup_{n\in\mathbb{N}} H_n \in \mathcal{F}$. By Lemma 4.29, we have $\mathbf{P}(H_n) = 0$, and therefore $\mathbf{P}(H) = 0$. Put

$$\tilde{v}_t(\omega) = \begin{cases} \lim_{\substack{q\in\mathbb{Q}_+ \\ q>t, q\downarrow t}} v_q(\omega), & \text{if } \omega \notin H_t, \\ 0, & \text{otherwise.} \end{cases}$$

If $\omega \notin H_t$, there is a $s > t$ such that for every $a < b$ we have $\omega \notin H_{s,a,b}$. Thus, the limit exists and is finite, and therefore $\tilde{v}$ is well defined. By the right-continuity of the filtration, $\tilde{v}$ is adapted. Let us next show that $\tilde{v}$ is right-continuous.

Assume $t \in \mathbb{R}_+$ and $\omega \in H_t$. Then for all $s > t$, $\omega \in H_t$, $\tilde{v}_s(\omega) = 0$, for $s \ge t$, and $\tilde{v}(\omega)$ is right-continuous at t. Let $\omega \notin H_t$. Since $H_t = \bigcap_{r>t} H_r$, there exists an $r_0 > t$ such that, for all $r \in]t, r_0]$, we have $\omega \notin H_r$. For any given $\epsilon > 0$ take $0 < \delta < r_0 - t$ such that $|\tilde{v}_t(\omega) - v_q(\omega)| \le \epsilon$ when $q \in \mathbb{Q}$, $s > t$, and $q - t < \delta$. Then, for $t < r < t + \delta$, we have

$$|\tilde{v}_t(\omega) - \tilde{v}_r(\omega)| = \lim_{\substack{q\in\mathbb{Q}_+ \\ q>r, q\downarrow r}} |\tilde{v}_t(\omega) - v_q(\omega)| \le \epsilon.$$

This means that $\tilde{v}(\omega)$ is right-continuous at t. Thus, all trajectories of $\tilde{v}$ are right-continuous, i.e., the process $\tilde{v}$ is right-continuous. It is left us to show that $\tilde{v}$ is a martingale.

Let $s < t$, $s, t \in \mathbb{R}_+$, and $(s_n), (t_n) \subset \mathbb{Q}_+$ be such that $s < s_n < t < t_n$, $s_n \downarrow s$ and $t_n \downarrow t$. For every $A \in \mathcal{F}_t$, we have

$$\int_A v_{s_n} \mathbf{P}(d\omega) = \int_A v_{t_n} \mathbf{P}(d\omega).$$

By Lemma 4.25, (v_{s_n}) and (v_{t_n}) are uniformly integrable, so by letting $n \to \infty$, we get

$$\int_A \tilde{v}_s \mathbf{P}(d\omega) = \int_A \tilde{v}_t \mathbf{P}(d\omega),$$

i.e., $\tilde{v}$ is a martingale. □

Theorem 4.32. (Doob's optional stopping theorem). *Let $v = (v_t)$, $t \in \overline{\mathbb{R}}_+$, be a right-continuous martingale, σ, τ be two stopping times, and $\sigma \leq \tau$. Then v_σ and v_τ are integrable, and*

$$\mathbf{E}[v_\tau | \mathcal{F}_\sigma] = v_\sigma \ \text{a.s.}. \tag{28}$$

Proof. Put $D_n = \{0, \frac{1}{2^n}, \frac{2}{2^n}, \cdots, \infty\}$, and

$$\sigma_n = \sum_{k=1}^{\infty} \frac{k}{2^n} 1_{\{\frac{k-1}{2^n} \leq \sigma < \frac{k}{2^n}\}} + \infty 1_{\{\sigma = \infty\}},$$

$$\tau_n = \sum_{k=1}^{\infty} \frac{k}{2^n} 1_{\{\frac{k-1}{2^n} \leq \tau < \frac{k}{2^n}\}} + \infty 1_{\{\tau = \infty\}}.$$

Then $v = (v_t)$, $t \in D_n$, is a martingale, σ_n and τ_n are stopping times, and $\sigma_n \downarrow \sigma$, $\tau_n \downarrow \tau$. By Theorem 4.27, we have that v_{σ_n} and v_{τ_n} are integrable, and

$$\mathbf{E}[v_{\tau_n} | \mathcal{F}_{\sigma_n}] = v_{\sigma_n} \ \text{a.s.}.$$

In particular, for every $A \in \mathcal{F}_\sigma \subset \mathcal{F}_{\sigma_n}$, we have

$$\int_A v_{\tau_n} \mathbf{P}(d\omega) = \int_A v_{\sigma_n} \mathbf{P}(d\omega).$$

Since $\lim_{n\to\infty} v_{\sigma_n} = v_\sigma$ and $\lim_{n\to\infty} v_{\tau_n} = v_\tau$, and by Theorem 4.6 (a), $v\sigma \in \mathcal{F}_\sigma$ and $v\tau \in \mathcal{F}_\tau$. To conclude that (28) holds it suffices to show that (v_{σ_n}) and (v_{τ_n}) are uniformly integrable. For $n \geq 1$, put

$$x_{-n} = v_{\sigma_n}, \text{ and } \mathcal{G}_{-n} = \mathcal{F}_{\sigma_n}.$$

Then $x = (x_n)$, $n \in \mathbb{N}_-$ is a martingale, and by Lemma 4.25, $(x_{-n}) = (v_{\sigma_n})$, $n \in \mathbb{N}$, is uniformly integrable. Similarly, one can show that (v_{τ_n}) is uniformly integrable. □

Theorem 4.33. (Doob's predictable stopping theorem). *Let $v = (v_t)$, $t \in \overline{\mathbb{R}}_+$, be a martingale, σ and τ be two stopping times, such that σ is predictable and $\sigma \le \tau$. Then*

$$\mathbf{E}[v_\tau|\mathcal{F}_{\sigma-}] = v_{\sigma-} \text{ a.s.}, \tag{29}$$

where $v_{\sigma-}$ is integrable and well defined for almost all $\omega \in \Omega$.

Proof. Since σ is predictable, by Corollary 4.24, there exists a sequence (σ_n) of stopping times foretelling σ. By Proposition 4.10, there exists a negligible set $N \subset \Omega$ such that

$$(\Omega \setminus N) \cap \mathcal{F}_{\sigma-} = (\Omega \setminus N) \cap \bigvee_n \mathcal{F}_{\sigma_n}.$$

By Theorem 4.32, we have

$$\mathbf{E}[v_\tau|\mathcal{F}_{\sigma_n}] = v_{\sigma_n} \text{ a.s.},$$

and applying Proposition 4.30, we obtain

$$\mathbf{E}[v_\tau|\mathcal{F}_{\sigma-}] = \lim_{n\to\infty} \mathbf{E}[v_\tau|\mathcal{F}_{\sigma_n}] = \lim_{n\to\infty} v_{\sigma_n} = v_{\sigma_-} \text{ a.s.}.$$

Let $c > 0$. By Theorem 4.32, for every n, we have

$$\mathbf{E}[|v_{\sigma_n\wedge c}|] = \mathbf{E}[v_{\sigma_n\wedge c}] + 2\mathbf{E}[v^-_{\sigma_n\wedge c}] \le \mathbf{E}[v_0] + 2\mathbf{E}[v^-_c] \le 3\sup_{t\ge 0}\mathbf{E}[|v_t|].$$

Letting $c \to \infty$, we get $\mathbf{E}[|v_{\sigma_n}|] \le 3\sup_{t\ge 0}\mathbf{E}[|v_t|]$, and therefore, v_{σ_-} is integrable. □

4.4 Projections of Processes

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space and $\mathbb{F} = (\mathcal{F}_t)$ be a filtration of $\mathcal{F}$ satisfying the usual conditions. Recall that if u and v are two stochastic process, then $u \le v$, if $\{(\omega, t) : u_t(\omega) \le v_t(\omega)\}$ is a $\mathbf{P}$-evanescent set.

Proposition 4.34. *Let $u = (u_t)$, $t \in \mathbb{R}$, and $v = (v_t)$, $t \in \mathbb{R}$, be two optional processes. Then $u \le v$ if and only if, for every bounded stopping time τ, $u_\tau \le v_\tau$ a.s.. If u and v are predictable, then it is sufficient that $u_\tau \le v_\tau$ a.s. holds for every bounded predictable stopping time τ.*

Proof. The necessity is obvious. We prove the sufficiency. Assume $u \le v$ does not hold, i.e., the set $\{(\omega, t) : u_t(\omega) > v_t(\omega)\}$ is not evanescent. Then, by Theorem 4.21, there exists a stopping time σ such that $\mathrm{Gr}(\sigma) \subset \{(\omega, t) : u_t(\omega) > v_t(\omega)\}$, and $\mathbf{P}(\{\tau < \infty\}) > 0$. Choose a constant $c > 0$ such that $\mathbf{P}(\sigma \le c) > 0$. Set $\tau = \sigma \wedge c$. Then τ is a bounded stopping time, and $u_\tau > v_\tau$ on $\{\sigma \le c\}$. This is a contradiction. We must have $u \le v$. For predictable u and v the contradiction follows from Theorem 4.21, and hence it suffices to show that the inequality holds for every bounded predictable stopping time τ. □

Proposition 4.35. *Let $u = (u_t)$, $t \in \mathbb{R}$, and $v = (v_t)$, $t \in \mathbb{R}$, be two optional processes. Assume that, for every stopping time τ, $u_\tau 1_{\{\tau<\infty\}}$ and $v_\tau 1_{\{\tau<\infty\}}$ are integrable. Then $u \leq v$ if and only if, for every stopping time τ, $\mathbf{E}[u_\tau 1_{\{\tau<\infty\}}] \leq \mathbf{E}[v_\tau 1_{\{\tau<\infty\}}]$. If u and v are predictable, then it is sufficient that $\mathbf{E}[u_\tau 1_{\{\tau<\infty\}}] \leq \mathbf{E}[v_\tau 1_{\{\tau<\infty\}}]$ holds for every predictable stopping time τ.*

Proof. The necessity is obvious. We prove the sufficiency. Assume $u \leq v$ does not hold, i.e., the set $\{(\omega, t) : u_t(\omega) > v_t(\omega)\}$ is not evanescent. Then, by Theorem 4.21, there exists a stopping time τ such that $\mathrm{Gr}(\tau) \subset \{(\omega, t) : u_t(\omega) > v_t(\omega)\}$, and $\mathbf{P}(\{\tau < \infty\}) > 0$. Then $\mathbf{E}[u_\tau 1_{\{\tau<\infty\}}] > \mathbf{E}[v_\tau 1_{\{\tau<\infty\}}]$ on $\{\sigma \leq c\}$. This is a contradiction. We must have $u \leq v$. For predictable u and v the contradiction follows from Theorem 4.21, and hence it suffices to show that the inequality holds for every predictable stopping time τ. □

Remark 4.36. In Proposition 4.34 and Proposition 4.35 one can replace "$\leq$" with "$=$".

Theorem 4.37. *Let $v = (v_t)$, $t \in \mathbb{R}$, be a measurable process.*

(a) *If $v_\tau 1_{\{\tau<\infty\}}$ is σ-integrable w.r.t. $\mathcal{F}_\tau$ for every stopping time τ, then there exists a unique optional process, denoted by ${}^o v$, such that for every stopping time τ and for every $G \in \mathcal{F}_\tau$ we have*

$$\int_G v_\tau 1_{\{\tau<\infty\}} \mathbf{P}(d\omega) = \int_G {}^o v_\tau 1_{\{\tau<\infty\}} \mathbf{P}(d\omega). \tag{30}$$

The process ${}^o v$ is called the optional projection of v.

(b) *If $v_\tau 1_{\{\tau<\infty\}}$ is σ-integrable w.r.t. $\mathcal{F}_{\tau-}$ for every predictable stopping time τ, then there exists a unique predictable process, denoted by ${}^p v$, such that for every predictable stopping time τ and for every $G \in \mathcal{F}_{\tau-}$ we have*

$$\int_G v_\tau 1_{\{\tau<\infty\}} \mathbf{P}(d\omega) = \int_G {}^p v_\tau 1_{\{\tau<\infty\}} \mathbf{P}(d\omega) \tag{31}$$

The process ${}^p v$ is called the predictable projection of v.

Proof. The uniqueness follows from Proposition 4.34 with Remark 4.36. We will show the existence.

(a) Assume first that $v = x 1_{[s,t[}$, where x is a bounded random variable and $0 \leq s \leq t \leq \infty$. Consider the martingale $(\mathbf{E}[x|\mathcal{F}_t])$. By Lemma 4.31, there exists a right continuous modification $u = (u_t)$ of $(\mathbf{E}[x|\mathcal{F}_t])$. Put ${}^o v = u 1_{[s,t[}$. It is clear that ${}^o v$ is optional. Moreover, by Theorem 4.32, ${}^o v$ satisfies (30), i.e., ${}^o v$ is the optional projection of v.

Assume now that u and v are bounded measurable processes. If ${}^o u$ and ${}^o v$ exist, then by σ-integrability of u and v, for any $\alpha, \beta \in \mathbb{R}$, we have ${}^o(\alpha u + \beta v) = \alpha {}^o u + \beta {}^o v$. Moreover, if $u \leq v$, then, by Proposition 4.34, ${}^o u \leq {}^o v$. Let $v^{(n)}$ be a monotone sequence of bounded measurable processes converging to v. Since, $v = \lim_{n\to\infty} v^{(n)}$ is bounded, by monotone convergence theorem, we have ${}^o v = \lim_{n\to\infty} {}^o v^{(n)}$. Now by

Theorem 2.8 with Remark 2.9, we conclude that the optional projection of every bounded process exists.

Suppose now that v is a non-negative measurable process such that $v_\tau 1_{\{\tau<\infty\}}$ is σ-integrable w.r.t. $\mathcal{F}_\tau$ for every stopping time τ. Put $v^{(n)} = v \wedge n$. By the preceding ${}^o v^{(n)}$ exists for every n, and, for every stopping time τ, the sequence $(v_\tau^{(n)} 1_{\{\tau<\infty\}})$ is increasing up to a null set. Let $G \in \mathcal{F}_\tau$. By monotone convergence theorem, we have

$$\int_G v_\tau 1_{\{\tau<\infty\}} \mathbf{P}(d\omega) = \lim_{n\to\infty} \int_G v_\tau^{(n)} 1_{\{\tau<\infty\}} \mathbf{P}(d\omega) = \lim_{n\to\infty} \int_G {}^o v_\tau^{(n)} 1_{\{\tau<\infty\}} \mathbf{P}(d\omega).$$

Put $u = \limsup_{n\to\infty} {}^o v^{(n)}$, and ${}^o v = u 1_{\{u<\infty\}}$. Then ${}^o v$ is a finite valued optional process, and for every stopping time τ, for every $G \in \mathcal{F}_\tau$, we have, by monotone convergence theorem

$$\begin{aligned}\int_G {}^o v_\tau 1_{\{\tau<\infty\}} \mathbf{P}(d\omega) = \int_G u_\tau 1_{\{\tau<\infty\}} \mathbf{P}(d\omega) &= \lim_{n\to\infty} \int_G {}^o v_\tau^{(n)} 1_{\{\tau<\infty\}} \mathbf{P}(d\omega) \\ &= \int_G v_\tau 1_{\{\tau<\infty\}} \mathbf{P}(d\omega),\end{aligned}$$

i.e., ${}^o v$ is the optional projection of v.

Assume finally that v is a measurable process satisfying the assumption of the theorem. Then v^+ and v^- satisfy the assumption of the theorem as well. Put ${}^o v = {}^o v^+ -^o v^-$. Then ${}^o v$ is an optional projection of v.

(b) Assume $v = x 1_{[s,t[}$, where x is a bounded random variable and $0 \le s \le t \le \infty$. Consider the martingale $(\mathbf{E}[x|\mathcal{F}_t])$. By Lemma 4.31, there exists a right continuous modification $u = (u_t)$ of $(\mathbf{E}[x|\mathcal{F}_t])$. Put ${}^p v = u_- 1_{[s,t[}$. It is clear that ${}^p v$ is predictable. Moreover, by Theorem 4.33, ${}^p v$ satisfies (30), i.e., ${}^p v$ is the optional projection of v. The rest of the proof is completely similar to that of (a). □

Remark 4.38. In Theorem 4.37, if we allow ${}^o v$ and ${}^p v$ be extended real valued, then the optional and predictable projection of an arbitrary measurable process exist uniquely.

Proposition 4.39. *Let u, $t \in \mathbb{R}$, be a measurable process and v, $t \in \mathbb{R}$, be an optional process. If ${}^o u$ exists, then ${}^o(uv)$ also exists and ${}^o(uv) = ({}^o u)v$. Respectively, if v is predictable and ${}^p u$ exists, then ${}^p(uv)$ also exists and ${}^p(uv) = ({}^p u)v$.*

Proof. A simple application of Proposition 2.3 (a). □

5 Optional and predictable projection of a normal integrand

The purpose of this chapter is to prove the existence and uniqueness of the optional and predictable projection of a normal integrand in continuous time. In discrete time, the optional and predictable projection are defined by taking conditional expectation on each time step. So, in discrete time, it is sufficient to show the existence and uniqueness of the conditional expectation, then the projections exist and are unique. For the normal integrand this is proven under various conditions; see e.g. Choirat et al. [2003].

5.1 Integrands in Continuous Time

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration of $\mathcal{F}$ satisfying the usual conditions. Let $\mathcal{S}$ be a sub-σ-algebra of $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$. A mapping $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \overline{\mathbb{R}}$ is said to be a *$\mathcal{S}$-integrand on $\mathbb{R}^d$*, if $f_t(\omega, x)$, as a function of (ω, t, x), is $\mathcal{S} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. If $\mathcal{S} = \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$, then we simply say that f is a measurable integrand on $\mathbb{R}^d$.

A subset E of $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d$ is said to be *evanescent* w.r.t. $\mathbf{P}$, if the projection $\pi(E)$ onto Ω is a $\mathbf{P}$-null set. Two measurable integrands f and g are said to be $\mathbf{P}$-indistinguishable, if $\{(\omega, t, x) : f_t(\omega, x) \neq g_t(\omega, x)\}$ is a $\mathbf{P}$-evanescent set; we abbreviate this by writing $f = g$. Later we will not distinguish two $\mathbf{P}$-indistinguishable measurable integrands, i.e., they are regarded as the same. Similarly, we write $f \leq g$, if $\{(\omega, t, x) : f_t(\omega, x) > g_t(\omega, x)\}$ is a $\mathbf{P}$-evanescent set.

We say that f is *of class $\mathcal{D}$*, if f is a measurable integrand on $\mathbb{R}^d$ and there exists an increasing sequence of bounded open sets (B_i), $i \geq 1$, that cover $\mathbb{R}^d$ and, for each $i \geq 1$, there exists a measurable process v^i such that, for every stopping time τ, $v^i_\tau 1_{\{\tau<\infty\}}$ is σ-integrable w.r.t. $\mathcal{F}_\tau$, and $f 1_{B_i} \geq v^i 1_{B_i}$.

Recall that a set-valued mapping $\Gamma : \Omega \rightrightarrows \mathbb{R}^d$ is measurable, if the preimage $\Gamma^{-1}(O) = \{\omega \in \Omega : \Gamma(\omega) \cap O \neq \emptyset\}$ is measurable for every open $O \subset \mathbb{R}^d$. The *epigraphical mapping* of f is defined as

$$\mathrm{epi} f : (\omega, t) \mapsto \{(x, \alpha) \in \mathbb{R}^d \times \mathbb{R} : f_t(\omega, x) \leq \alpha\}.$$

We say that an integrand f is *normal*, if its epigraphical mapping is closed valued and measurable. A simple example of normal integrands are *Carathéodory* integrands, which are continuous for every $(\omega, t) \in \Omega \times \mathbb{R}_+$ and measurable for every $x \in \mathbb{R}^d$. See [Rockafellar et al., 1998] Example 14.29. We say that an integrand f is *lower semicontinuous*, if for every $(\omega, t, x) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^d$, for every (y_k) converging to x, we have

$$\liminf_{y_k \to x} f(\omega, y_k) \geq f(\omega, x).$$

A normal $\mathcal{S}$-integrand is always lower semicontinuous and $\mathcal{S}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable, but the reverse statement holds only when $\mathcal{S}$ is complete with respect to some σ-finite measure. See [Rockafellar et al., 1998] Corollary 14.34.

In Chapter 3, we established that $\mathcal{S} \subset \mathcal{A}(\mathcal{S})$ always, and $\mathcal{S} \supset \mathcal{A}(\mathcal{S})$ if and only if $\mathcal{S}$ is complete with respect to some σ-finite measure. Take any analytic set $A \in \mathbb{R}_+$, which is not Borel, then $\Omega \times A$ is analytic, but $\Omega \times A \notin \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$. We conclude that our underlying measurable space $(\Omega \times \mathbb{R}_+, \mathcal{S})$, where $\mathcal{S} \subset \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$, is not complete with respect to any σ-finite measure, and therefore lower semicontinuity and measurability do not imply normality.

Lemma 5.1. *Let f be a lower semicontinuous $\mathcal{S}$-integrand on $\mathbb{R}^d$. Then the following statements are equivalent:*

(a) *f is a normal $\mathcal{S}$-integrand on $\mathbb{R}^d$,*

(b) *The process $(\omega, t) \mapsto \inf_{x\in B} f_t(\omega, x)$ is $\mathcal{S}$-measurable for every open $B \subset \mathbb{R}^d$.*

Proof. [Rockafellar et al., 1998] Proposition 14.40. □

We say that an integrand f is *k-Lipschitz*, for $k \geq 0$, if for every $(\omega, t) \in \Omega \times \mathbb{R}_+$, $f_t(\omega, \cdot) \equiv \infty$, or, for all $x, y \in \mathbb{R}^d$, we have

$$|f_t(\omega, x) - f_t(\omega, y)| \leq kd(x, y).$$

It is obvious that, if f is k-Lipschitz, then it is continuous, and therefore lower semicontinuous. Since we identify indistinguishable measurable integrands, it is sufficient that the given conditions for normality, lower semicontinuity, and k-Lipschitzianity are satisfied outside an evanescent set.

5.2 Projections of Integrands

Let f be an integrand and v a stochastic process. The extended real valued process $f(v) : (\omega, t) \mapsto f_t(\omega, v_t(\omega))$ is called the *composite map process* of f and v.

Definition 5.2. Let f be a measurable integrand on $\mathbb{R}^d$. We say that ${}^o f$, an $\mathcal{O}$-integrand on $\mathbb{R}^d$, is an *optional projection* of f, if

$$({}^o f)(v) = {}^o f(v) \tag{32}$$

for every optional process v. Similarly, ${}^p f$, a $\mathcal{P}$-integrand on $\mathbb{R}^d$, is called a *predictable projection* of f, if

$$({}^p f)(v) = {}^p f(v) \tag{33}$$

for every predictable process v.

Proposition 5.3. *Let f and g be $\mathcal{O}$-integrands on $\mathbb{R}^d$. Then $f \leq g$ if and only if, for every optional process v, every stopping time τ and every $G \in \mathcal{F}_\tau$, the following inequality holds*

$$\int_G f_\tau(v_\tau) 1_{\{\tau < \infty\}} \mathbf{P}(d\omega) \leq \int_G g_\tau(v_\tau) 1_{\{\tau < \infty\}} \mathbf{P}(d\omega). \tag{34}$$

If f and g are $\mathcal{P}$-integrands, then it is sufficient that (34) *holds for every predictable process v, every predictable stopping time τ, and every $G \in \mathcal{F}_{\tau-}$.*

Proof. The necessity is obvious. We show the sufficiency. Put

$$\Lambda = \{(\omega, t, x) : f_t(\omega, x) > g_t(\omega, x)\}.$$

If Λ is non-evanescent, then we may choose constants $a, b, c \in \mathbb{R}$ such that the set

$$\tilde{\Lambda} = \{(\omega, t, x) : f_t(\omega, x) > a > b > g_t(\omega, x), |x| \leq c\}$$

is non-evanescent. Consider the set-valued mapping

$$\Gamma(\omega, t) = \{x \in \mathbb{R}^d : (\omega, t, x) \in \tilde{\Lambda}\}.$$

Let $A = \mathrm{dom}(\Gamma)$. By Theorem 3.3 we have $A \in \mathcal{A}(\mathcal{O})$. Further, by Theorem 3.10 $\mathcal{A}(\mathcal{O}) \subset \widehat{\mathcal{O}}$, and hence $A \in \widehat{\mathcal{O}}$. Let π denote the projection onto Ω and let $\epsilon > 0$.

By Lemma 4.19 there exists an optional set B such that $B \subset A$ and $\mathbf{P}(\pi(B)) > \mathbf{P}(\pi(A)) - \epsilon/2$. By Theorem 4.20 there exists a stopping time τ such that $\mathrm{Gr}(\tau) \subset B$ and $\mathbf{P}(\{\tau < \infty\}) > \mathbf{P}(\pi(B)) - \epsilon/2$. Since $\pi(A) = \pi(\tilde{\Lambda})$, for any given $\epsilon > 0$, we have

$$\mathbf{P}(\{\tau < \infty\}) > \mathbf{P}(\pi(\tilde{\Lambda})) - \epsilon.$$

On the other hand, $\mathrm{Gr}(\Gamma)$ is equal $\tilde{\Lambda}$, so, by Theorem 3.12, there exists a $\widehat{\mathcal{O}}$-measurable selection $\widehat{v}$ of Γ. Define a measure μ on $\mathcal{O}$ by

$$\mu(C) = \mathbf{P}(\{\omega : (\omega, \tau(\omega)) \in C\}).$$

We have that B is the support of μ, i.e., μ vanishes outside B, and that $\mu(B) = \mathbf{P}(\{\tau < \infty\}) = \mathbf{P}(\pi(B))$. Since $\mathcal{O}$ and $\widehat{\mathcal{O}}$ differ only in the μ-zero sets, there exists an $\mathcal{O}$-measurable v such that

$$\mu(\{(\omega, t) : v_t(\omega) \neq \widehat{v}_t(\omega)\}) = 0.$$

Put $G = \{\omega : (\omega, \tau(\omega)) \in B\}$. Then $1_G = 1_B(\tau) 1_{\{\tau < \infty\}}$, and G is $\mathcal{F}_\tau$-measurable by Theorem 4.6 (a). If Λ is not evanescent, we may choose $\epsilon > 0$ such that

$$\int_G f_\tau(\omega, v_\tau) 1_{\{\tau < \infty\}} \mathbf{P}(d\omega) > \int_G g_\tau(\omega, v_\tau) 1_{\{\tau < \infty\}} \mathbf{P}(d\omega).$$

This is a contradiction. Hence Λ is evanescent, i.e., $f \leq g$. The proof for predictable integrands is similar; proceed as above, but use Theorem 4.21 instead of Theorem 4.20. $\square$

Corollary 5.4. *Let f and g be measurable integrand on $\mathbb{R}^d$ such. If $f \leq g$, then ${}^o f \leq {}^o g$ and ${}^p f \leq {}^p g$ whenever the projections exist.*

Proof. For every stopping time τ, for all $G \in \mathcal{F}_\tau$ and for every optional process v, we have

$$\begin{aligned}\int_G {}^o f_\tau(\omega, v_\tau) 1_{\{\tau < \infty\}} \mathbf{P}(d\omega) &= \int_G f_\tau(\omega, v_\tau) 1_{\{\tau < \infty\}} \mathbf{P}(d\omega) \\ &\leq \int_G g_\tau(\omega, v_\tau) 1_{\{\tau < \infty\}} \mathbf{P}(d\omega) \\ &= \int_G {}^o g_\tau(\omega, v_\tau) 1_{\{\tau < \infty\}} \mathbf{P}(d\omega).\end{aligned}$$

Hence, by Proposition 5.3, ${}^o f \leq {}^o g$. By doing the obvious modifications to the proof, we get ${}^p f \leq {}^p g$. $\square$

Proposition 5.5. *Let f and g be $\mathcal{O}$-integrands on $\mathbb{R}^d$. Then $f \leq g$ if and only if, for every optional process v and every bounded stopping time τ, one has*

$$f_\tau(v_\tau) \leq g_\tau(v_\tau) \text{ for almost all } \omega \in \Omega. \tag{35}$$

If f and g are $\mathcal{P}$-integrands, then it is sufficient that (35) holds for every predictable process v and every bounded predictable stopping time τ.

Proof. Proceed as in the proof of Proposition 5.3, only noting that you may choose $C > 0$ such that $P(\tau \le C) > 0$, and define $\sigma = \tau \wedge C$, then $f_\sigma(\omega, v_\sigma) > g_\sigma(\omega, v_\sigma)$ on $\{\sigma \le C\}$. If we have predictable integrands, then τ is predictable, and by the same token, σ is predictable. □

Remark 5.6. In Proposition 5.3, Corollary 5.4 and Proposition 5.5, one can replace "$\le$" with "$=$".

Theorem 5.7. *Let f be a measurable integrand on $\mathbb{R}^d$. Then there exists a unique optional projection ${}^o f$ and a unique predictable projection ${}^p f$.*

Proof. The uniqueness is an immediate consequence of Proposition 5.3. To prove the existence, we apply a monotone class argument. We treat only the optional case; the proof for the predictable projection is the same.

Consider $F = A \times B$, where $A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ and $B \in \mathcal{B}(\mathbb{R}^d)$, and assume that the integrand is defined as $f(\omega, t, x) = 1_F(\omega, t, x)$. Then the optional projection is well defined. Indeed, for all stopping times τ, for all $G \in \mathcal{F}_\tau$ and for all optional processes v, we have

$$\begin{aligned}
\int_G 1_A(\omega,\tau)1_B(v_\tau)1_{\{\tau<\infty\}}\mathbf{P}(d\omega) &= \int_G 1_A(\omega,\tau)1_B(v_\tau 1_{\{\tau<\infty\}})1_{\{\tau<\infty\}}\mathbf{P}(d\omega) \\
&= \int_G 1_A(\omega,\tau)1_{(v_\tau 1_{\{\tau<\infty\}})^{-1}(B)}1_{\{\tau<\infty\}}\mathbf{P}(d\omega) \\
&= \int_G {}^o(1_A(\omega,\tau)1_{(v_\tau 1_{\{\tau<\infty\}})^{-1}(B)}1_{\{\tau<\infty\}})\mathbf{P}(d\omega) \\
&= \int_G 1_{(v_\tau 1_{\{\tau<\infty\}})^{-1}(B)}\ {}^o(1_A(\omega,\tau)1_{\{\tau<\infty\}})\mathbf{P}(d\omega).
\end{aligned}$$

The last equality follows from the Corollary 4.6 and Proposition 4.39. This shows that ${}^o f$ is defined by ${}^o f(\omega, t, x) = {}^o 1_A(\omega, t)1_B(x)$, which clearly is an $\mathcal{O}$-integrand on $\mathbb{R}^d$.

The rest of the proof is completely to corresponding part of the proof of existence of conditional expectation of an integrand; see Choirat et al. [2003]. In Theorem 2.8, let $\mathcal{C} = \big(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)\big) \times \mathcal{B}(\mathbb{R}^d)$ and $\mathcal{V}$ be the class of functions f for which ${}^o f$ exists. Above we showed that the condition (a) in Theorem 2.8 is satisfied. Since we allow the value ∞, it easy to see that the condition (b) is satisfied. The condition (c) holds true by the monotone convergence theorem. When it comes to the condition (d), the problem may arise when neither f or g is integrable, but remember that we made the convention that the integral is defined to be ∞ whenever positive part is not integrable. This with the convention $\infty - \infty = \infty$ guarantees that the condition (d) always holds. By Theorem 2.8, there exists an optional projection for every measurable integrand. □

Lemma 5.8. *Let f be a normal integrand on $\mathbb{R}^d$. Assume that $f \in \mathcal{D}$ and let B be a member of the open cover. For every $k \ge 1$, $\omega \in \Omega$, $t \in \mathbb{R}_+$ and $x \in B$, put*

$$f_t^k(\omega, x) = \inf_{y \in B}\{f_t(\omega, y) + kd(x, y)\}. \tag{36}$$

Then

(a) f^k *is a measurable k-Lipschitz integrand on B;*

(b) $(f^k)_{k\geq 1}$ *is increasing and converges to f as $k\to\infty$;*

(c) ${}^o(f^k)$ *is a k-Lipschitz $\mathcal{O}$-integrand on B;*

(d) ${}^p(f^k)$ *is a k-Lipschitz $\mathcal{P}$-integrand on B.*

Proof. If $f_t(\omega,\cdot)\equiv\infty$, then $f_t^k(\omega,\cdot)\equiv\infty$, and the assertions (a), (b), (c) and (d) hold. On the other hand, if $f_t(\omega,\cdot)$ is finite somewhere on B, then $f_t^k(\omega,\cdot)$ is finite everywhere on B. Hence, without a loss of generality, we assume $f_t^k(\omega,\cdot)$ is finite everywhere.

(a) Since f is normal, a mapping $(\omega,t,y)\mapsto f_t(\omega,y)+kd(x,y)$ is normal for every $x\in B$, and, by Lemma 5.1,

$$(\omega,t)\mapsto f_t^k(\omega,x)=\inf_{y\in B}\{f_t(\omega,y)+kd(x,y)\}$$

is a measurable process for every $x\in B$. So, by Lemma 5.1, if we show that f^k is k-Lipschitz, the measurability follows. Let $f_t^k(\omega,x)\geq f_t^k(\omega,y)$ and $z^*=\arg\inf_{z\in B}\{f(\omega,z)+kd(y,z)\}$. We have

$$\begin{aligned}|f_t^k(\omega,x)-f_t^k(\omega,y)|&=\inf_{z\in B}\{f_t(\omega,z)+kd(x,z)\}-f_t(\omega,z^*)-kd(y,z^*)\\&\leq f_t(\omega,z^*)+kd(x,z^*)-f_t(\omega,z^*)-kd(y,z^*)\leq kd(x,y).\end{aligned}$$

Hence, f^k is k-Lipschitz.

(b) Let $y_k=\arg\inf_{y\in B}\{f_t(\omega,y)+kd(x,y)+1/k\}$ for every $k\geq 1$. The sequence is increasing, $f_t^k(\omega,x)\leq f_t(\omega,x)$ for every $k\geq 1$, and $y_k\to x$ as $k\to\infty$. Hence,

$$\liminf_{y_k\to x}\{f_t(\omega,y_k)+kd(x,y)\}\leq\lim_{k\to\infty}f_t^k(\omega,x)\leq f_t(\omega,x).$$

On the other hand, by the lower semicontinuity of f, we have

$$\liminf_{y_k\to x}\{f_t(\omega,y_k)+kd(x,y_k)\}\geq\liminf_{y_k\to x}f_t(\omega,y_k)\geq f_t(\omega,x).$$

Hence, $f_t^k(\omega,x)\uparrow f_t(\omega,x)$.

(c) This is a proof by contradiction. Assume that there exists a non-evanescent set $A\subset\Omega\times\mathbb{R}_+$ such that ${}^of^k$ is not k-Lipschitz. Since ${}^of$ is optional, the set A is optional. By Theorem 4.20, there exists a stopping time σ, with $\mathbf{P}(\{\sigma<\infty\})>0$, such that $\mathrm{Gr}(\sigma)\subset A$. Choose a constant $C>0$ such that for $\tau=\sigma\wedge C$, we have $\mathbf{P}(\{\tau<\infty\})>0$ and $\mathrm{Gr}(\tau)\subset A$. For every $x\in B$, there exists a negligible set N_x such that ${}^of_\tau^k(\omega,x)=\mathbf{E}[f_\tau^k(\cdot,x)|\mathcal{F}_\tau](\omega)$ for every $\omega\notin N_x$. Put $N=\bigcup_{x\in B\cap\mathbb{Q}^d}N_x$. Let $x,y\in B\cap\mathbb{Q}^d$. Then, for every $\omega\notin N$, we have ${}^of_\tau^k(\omega,x)=\infty$ if and only if ${}^of_\tau^k(\omega,y)=\infty$. Hence, without a loss of generality, we may assume that $f_\tau^k(\omega,x)<\infty$ for every $x\in B\cap\mathbb{Q}^d$. For all $\omega\in\Omega\setminus N$ and $x,y\in B\cap\mathbb{Q}^d$, we have

$$|{}^of_\tau^k(\omega,x)-{}^of_\tau^k(\omega,y)|\leq\mathbf{E}[|f_\tau^k(\cdot,x)-f_\tau^k(\cdot,y)||\mathcal{F}_\tau](\omega)\leq kd(x,y).\tag{37}$$

For each $\omega \in \Omega \setminus N$ and $x \in B$, put $h_\tau(\omega, x) = \lim_{n\to\infty} {}^o f^k_\tau(\omega, x_n)$, where $(x_n) \subset \mathbb{Q}^d$ is an arbitrary sequence converging to x. Then $h_\tau : (\Omega \setminus N) \times B \to \mathbb{R}$ is a well defined mapping, and for all $x, y \in B$, we have

$$|h_\tau(\omega, x) - h_\tau(\omega, y)| \leq kd(x, y). \tag{38}$$

It is left us to verify that $h_\tau(\omega, x) = [f^k_\tau(\cdot, x)|\mathcal{F}_\tau](\omega)$ a.s.. Let $(x_n) \subset B \cap \mathbb{Q}^d$ be a sequence converging to $x \in B$, and $G \in \mathcal{F}_\tau$. We have

$$\int_G h_\tau(\omega, x_n)\mathbf{P}(d\omega) = \int_G {}^o f^k_\tau(\omega, x_n)\mathbf{P}(d\omega) = \int_G f^k_\tau(\omega, x_n)\mathbf{P}(d\omega). \tag{39}$$

Let $n \to \infty$ in (39). By (37) and (38), we get

$$\int_G h_\tau(\omega, x)\mathbf{P}(d\omega) = \int_G f^k_\tau(\omega, x)\mathbf{P}(d\omega).$$

Hence, $h_\tau(\omega, x) = [f^k_\tau(\cdot, x)|\mathcal{F}_\tau](\omega)$ a.s.. This is a contradiction. We must have that ${}^o f^k$ is k-Lipschitz.

(d) The proof is essentially the same as the proof of the previous assertion (c). Conclude from Theorem 4.21 that you may fix a bounded predictable stopping time, then proceed as in the proof of the assertion (c) and show that the integrand is k-Lipschitz. □

Theorem 5.9. *Let $f \in \mathcal{D}$. If f is normal, so are ${}^o f$ and ${}^p f$.*

Proof. Let $(B_i)_{i\geq 1}$ be the open cover associated to f being a member of class $\mathcal{D}$. For every $i \geq 1$, $k \geq 1$, $\omega \in \Omega$, $t \in \mathbb{R}_+$ and $x \in B_i$, put

$$f^{i,k}_t(\omega, x) = \inf_{y\in B_i} \{f_t(\omega, y) + kd(x, y)\}.$$

Then, by Lemma 5.8 (a), for every $i \geq 1$, for every $k \geq 1$, $f^{i,k}$ is k-Lipschitz on B_i. By Lemma 5.8 (b), for every $i \geq 1$, $f^{i,k} \uparrow f^i$. By Corollary 5.4, optional projection preserves order, so, for every $i \geq 1$, the sequence $({}^o f^{i,k})_{k\geq 1}$ is an increasing sequence, and by Lemma 5.8 (c), each element of the sequence is a k-Lipschitz $\mathcal{O}$-integrand on B_i. Since $f \in \mathcal{D}$, we may apply monotone convergence theorem on each B_i: for every stopping time τ, for every optional process v, we have

$${}^o(\sup_k f^{i,k}_\tau(\omega, v_\tau)) = \mathbf{E}[\sup_k f^{i,k}_\tau(\omega, v_\tau)|\mathcal{F}_\tau] = \sup_k \mathbf{E}[f^{i,k}_\tau(\omega, v_\tau)|\mathcal{F}_\tau] = \sup_k {}^o f^{i,k}_\tau(\omega, v_\tau).$$

Hence, ${}^o(f^i) = \sup_k {}^o(f^{i,k})$. Since ${}^o(f^{i,k})$ are normal on the respective B_i, so is ${}^o(f^i)$. Since (B_i), $i \in \mathbb{N}$, is an increasing sequence of open sets that cover $\mathbb{R}^d$, ${}^o f$ is normal on $\mathbb{R}^d$. The proof that ${}^p f$ is normal is the same; use the assertion (d) of Lemma 5.8 instead of (c). □

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